An Exact Algorithm for the Statistical Shortest Path Problem

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Outline

- Motivation
- Statistical shortest path (SSP) problem
- Our exact algorithm for SSP problem
- Applications
  - Maze Routing
  - Timing Analysis
  - Buffer Insertion
Why Statistical Methods?

- Intra-die variations become dominant
- Corner-based design flow leads to over design or yield loss
- Statistical methods are needed not only in simulation but also in design tools.

Temperature Variation in Cell Processor
Dac C. Pham, et al. ISSCC05
Variations, Performance and Yield

- Variation sources
  - Process variations
    - Gate length variation
    - Geometric variation in interconnection wires
  - Temperature variations
  - Supply voltage variations
- Statistical models for circuits have been proposed
- New algorithm considering variations are needed for performance/yield optimization
Statistical Model for Variations

- Use mean $\mu$ and variance $\sigma^2$ to capture the random property of variations
- Exact for Gaussian, uniform, binominal, exponential distributions and etc.
- Good approximation for arbitrary random variables
Mean and variance are additive, but not the standard deviation $\sigma$

Recall the Chebyshev’s Inequality:

$$P(|X - \mu| \leq k\sigma) > 1 - \frac{1}{k^2}$$

The cost function $\mu + k\sigma$ is important to yield optimization

$\sigma$ not additive presents difficulties in solving statistical graph problems
Statistical Shortest Path Problem

- Edge weights are random variables
- To find a path with minimum $\mu + \Phi(\sigma^2)$ value
- Existing methods cannot solve this problem

Edge weight: (mean, variance)
From Deterministic to Statistical

<table>
<thead>
<tr>
<th>Deterministic</th>
<th>Statistical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edge weight $w$</td>
<td>Edge weight $(\mu, \sigma^2)$</td>
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<tr>
<td>$w$ is additive</td>
<td>$\mu, \sigma^2$ are additive</td>
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<tr>
<td>Path weight $\Sigma w$</td>
<td>Path weight $(\mu_p, \sigma_p^2)$</td>
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<tr>
<td>Minimize $\Sigma w$</td>
<td>Minimize $\mu_p + \Phi(\sigma_p^2)$</td>
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Statistical Shortest Path Problem

- Given a directed graph $G$
  - Not necessarily a DAG
- Find a path $p$ from source vertex $s$ to sink vertex $t$ such that
  - $\mu_p + \Phi(\sigma_p^2)$ is minimized
  - Path weight of $p$ is a random variable with mean $\mu_p$ and variance $\sigma_p^2$
Practical Observations for EDA problems

- $\mu, \sigma^2$ are additive
- For yield optimization problems
  - $\sigma^2$ is bounded
  - $\sigma^2$ can be discretized without introducing much error
- We may assume the variance $\sigma^2$ of path weight are integers upper bounded by $B$, i.e., $\sigma^2 \leq B$
Algorithm for Solving SSP Problem

- Vertex splitting for $\mu$, $\sigma^2$
- Graph expansion to generate a new graph $G'$
- $G'$ has real numbers as its edge weights
- Each vertex $u$ in $G$ is split into a set of vertices in $G'$: $\{u_1, u_2, \ldots, u_B\}$
Graph Expansion – Source Node

- From source to other vertices
- Only expand vertex a
- Each new vertex $a_i$ corresponds to a with variance $i$
- Edge weight is $\mu$
Graph Expansion – Internal Nodes

- Assuming vertex $u$ is already split
- Its neighbor $v$ will be also split
- Edges are connected according to $\sigma^2$ of path weight
- Edge weight are $\mu$
Graph Expansion – Sink Node

- Original sink node is already split according to previous steps
- Add a super sink node $t'$
- Edge weight for edge $ti$ to $t'$ is $\Phi(i)$
- Note that any path from source to $ti$ has variance equals to $i$
From Arbitrary Graph to DAG

- There will be no loop in expanded graph since $\sigma^2 > 0$
SSP Algorithm

- The expanded graph $G'$ is a DAG
- Shortest path in $G'$ can be found by existing deterministic shortest path algorithms for DAG
- This path corresponds to a path in $G$ that minimizes $\mu_p + \Phi(\sigma_p^2)$
- Time complexity is $O(B(V+E))$
Improvement

- Only split a vertex whenever it is necessary; don’t split all vertices
- Remove redundant vertices during splitting
  - If paths have same variance, then the one with larger mean is redundant
  - If $\Phi(\sigma_p^2)$ is a monotonically increasing function, paths with larger mean and variance are redundant
Example

Edge weight: (mean, variance)

\[ \Phi(x) = 3\sqrt{x} \]
SSP Algorithm Improved

- Much less vertices are generated
  - 100 vertices needed for previous example with original approach
  - 10 vertices used with improved algorithm
- Expand graph simultaneously with searching the shortest path
- Much faster with less memory requirement
EDA applications

- Maze Routing
- Timing Analysis
- Buffer Insertion
Maze Routing

- Timing-driven maze routing
- Process Variations
  - Systematic variations
  - Random variations
  - Temperature variations
- Find the shortest path to improve the performance
Maze Routing

No Variations considered

Variations considered
Timing Analysis

- Find the longest delay path considering intra-die variations
- Large circuits with several logic levels
  - Gaussian distribution for the path delay
  - $\mu_p + 3\sigma_p$ is used to measure the timing-yield
- Our algorithm can also find the (path) candidates with longest delay
Timing Analysis

- ISCAS benchmarks
- Cell delays are not necessarily Gaussian
- 40X—1000X runtime improvement over Monte Carlo simulation
- Very little error compare to Monte Carlo method
Buffer Insertion

- Buffer insertion in 2-pin net can be formulated into shortest path problem
- With variations from both devices and interconnections, it should be formulated into statistical shortest path problem
- Our algorithm can solve this buffer insertion with variations consideration
Buffer Insertion

- Graph based approach
- Formulated as a shortest path problem
Conclusion

- Exact algorithm to solve the statistical shortest path problem
- Arbitrary graph, arbitrary cost function $\Phi$
- Efficient implementation
- Can be used in varieties of applications in nanometer design