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Distribution Arithmetic for Stochastical Analysis

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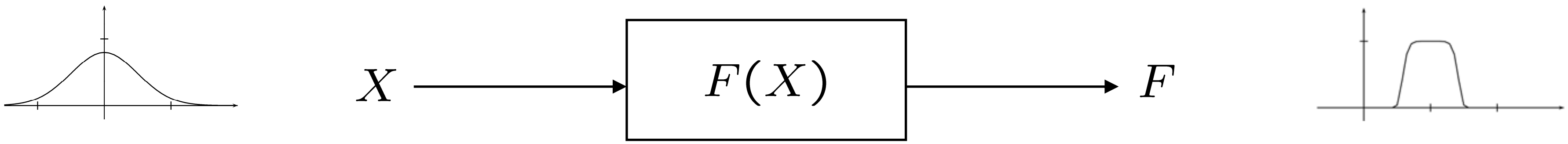
Overview

- Stochastical Analysis
- Moments
- Ideas behind the Distribution Arithmetic
- Distribution Arithmetic
- Results

Parameters Vary – Performance Values Vary

- Parameters
  - Process Parameters (e.g. V_{th}, t_{ox})
  - Design Parameters (e.g. W, L)
  - Operational Parameters: (e.g. V_{dd}, Temp)
- Reasons
  - Manufacturing variability, environment change, operational conditions, …
- Impact
  - Performance values vary: delay, gain, …

Parameters and Performance Values are Random Values (RVs)



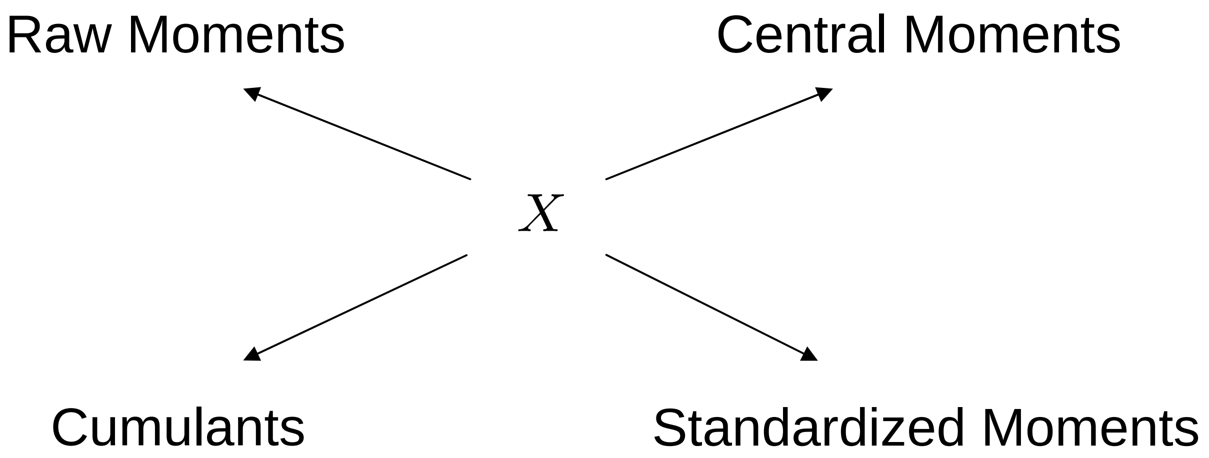
Parameters
(e.g. R_{square})

Performance Value
(e.g. Delay)

- Random values are distributed values.
- Representations:
  – Probability/Cumulative density function (PDF/CDF)
  – Moments

Moments

- Moments are integral properties of a distribution
- Examples: Mean, Variance, Skewness, Kurtosis



Circuit Analysis

- Task: Given the distributions of X, determine the distribution of F(X).
- Known methods
  - Monte Carlo
    - Many samples necessary
  - Range Arithmetic (Interval Arithmetic, Affine Arithmetic)
    - Only ranges are calculated
  - Response Surface Methods
    - Approximation of F(X) must exist

- How can we calculate with random variables directly?

Challenges of Building a Distribution Arithmetic

- Description of RV's distributions
- Consideration of correlations between RVs
- Procedures for mathematical operations on RVs
  - Unary Operations: -, sqrt, log, …
  - Binary Operations: +, *, …
- Determination of RV's distribution properties

Basic Ideas Behind The Distribution Arithmetic

1. Each RV is a linear combination of powers of „Initial Random Values“ (IRVs).
2. Mathematical operations are approximated by polynomials (Taylor-Series).
3. Moments of any RV can be determined by a weighted sum of given IRV moments.
4. Semi-symbolic implementation for fast evaluation

Initial Random Values (IRVs)

- Each independent reason for variability is represented by an IRV Δ_i
- Without loss of generality:
  Expected value $\mathrm{E}(\Delta_i)$ of each IRV is 0.
- IRVs can be arbitrarily distributed.
- Only moments of IRVs must be known
  (or taken from tables).
- IRV symbols are never replaced by values.

RV Representation: Examples

$$W = w_o + \sigma_W \Delta_1$$

W	:	Random variable for a width
w_o	:	Nominal value
σ_W	:	Standard deviation (if Δ_1 is standard normal distributed)
Δ_1	:	Initial random value

A more general example:

$$\begin{aligned} X = \; & x_{2,0}\Delta_1^2 \\ & + x_{1,0}\Delta_1 \quad + x_{1,1}\Delta_1\Delta_2 \\ & + x_{0,0} \qquad + x_{0,1}\Delta_2 \qquad + x_{0,2}\Delta_2^2 \end{aligned}$$

General RV Representation

$$X = \sum_{\vec{i} \in D} \left(x_{i_1,\ldots,i_n} \prod_{j=1}^{n} \Delta_j^{i_j} \right)$$

with

$$D = \left\{ \vec{i} \in \mathbb{N}^n \middle| \sum_{j=1}^{n} i_j \leq l \right\}$$

Δ_i : IRVs (wlog. $\mathrm{E}(\Delta_i) = 0$)

n : Number of IRVs

l : DA's order

Unary Operations

- Nonlinear functions T are approximated by a Taylor Series:

$$T(X) \approx \sum_{i=0}^{l} \frac{T_{X^i}(x_0)}{i!}(\Delta X)^i$$

with

$$\begin{aligned} T_{X^i} &:= \frac{\partial^i}{\partial X^i} T(x_0) \\ \Delta X &:= (X - x_0) \end{aligned}$$

Example: 2nd Order Operation

$$
\begin{aligned}
X = \; & x_{2,0}\Delta_1^2 \\
& + x_{1,0}\Delta_1 \quad + x_{1,1}\Delta_1\Delta_2 \\
& + x_{0,0} \quad\;\; + x_{0,1}\Delta_2 \quad\;\; + x_{0,2}\Delta_2^2
\end{aligned}
$$

$$
\begin{aligned}
T(X) \approx \; & T(x_0) \\
& + T^{'}(x_0)(X - x_0) \\
& + \frac{T^{''}(x_0)}{2!}(X - x_0)^2
\end{aligned}
$$

$$
\begin{aligned}
T = \; & t_{4,0}\Delta_1^4 \\
& + t_{3,0}\Delta_1^3 \quad + t_{3,1}\Delta_1^3\Delta_2 \\
& + t_{2,0}\Delta_1^2 \quad + t_{2,1}\Delta_1^2\Delta_2 \quad + t_{2,2}\Delta_1^2\Delta_2^2 \\
& + t_{1,0}\Delta_1 \quad + t_{1,1}\Delta_1\Delta_2 \quad + t_{1,2}\Delta_1\Delta_2^2 \quad + t_{1,3}\Delta_1\Delta_2^3 \\
& + t_{0,0} \quad + t_{0,1}\Delta_2 \quad + t_{0,2}\Delta_2^2 \quad + t_{0,3}\Delta_2^3 \quad + t_{0,4}\Delta_2^4
\end{aligned}
$$

neglected

Binary Operations

- Sum: Simple adding of corresponding coefficients
- Product: Expanding and then neglecting of higher IRV powers
- Others are traced back to +, * and unary operations:

$$
\begin{aligned}
\frac{x}{y} &= x(y^{-1}) \\
x^y &= e^{(y \log x)}
\end{aligned}
$$

Moments Determination

$$m_{X,k} = \cdots + C\,\mathsf{E}(\prod_{j=1}^{n} \Delta_j^{i_j}) + \cdots$$

is simple using given IRV moments:

$$\begin{aligned} \mathsf{E}(\Delta_i^k) &= m_{\Delta_i,k} \\ \mathsf{E}(\Delta_i^k \Delta_j^l) &= m_{\Delta_i,k}\, m_{\Delta_j,l} \end{aligned}$$

$$\begin{aligned} m_{\Delta_i,0} &= 1 \\ m_{\Delta_i,1} &= 0 \end{aligned}$$

$m_{\Delta_i,k}$: kth raw moment of Δ_i

Semi-Symbolic Implementation

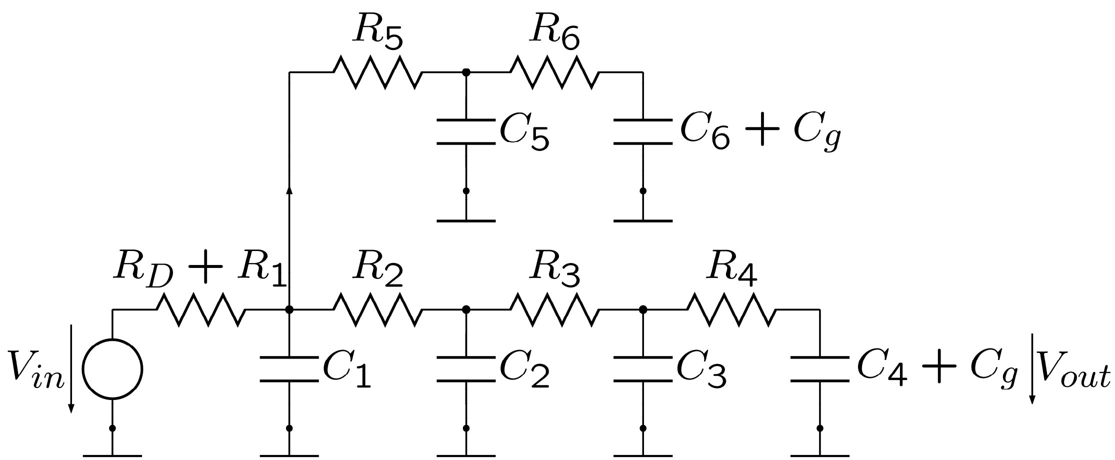
- C++ using operator overloading
- Delta symbols are not stored for each variable.
  Only coefficients are stored in arrays:

$$X = x_{2,0}\Delta_1^2 + x_{1,0}\Delta_1 + x_{1,1}\Delta_1\Delta_2 + x_{0,0} + x_{0,1}\Delta_2 + x_{0,2}\Delta_2^2$$

$$\begin{pmatrix} x_{2,0} & & \\ x_{1,0} & x_{1,1} & \\ x_{0,0} & x_{0,1} & x_{0,2} \end{pmatrix}$$

- Delta symbol powers arise from position
- Implemented Functions
  - All standard math-operations (+,*,/, log, sqrt, ...)
  - Moment determination (Variance, Skew, Kurtosis, ...)
  - Correlation coefficient determination, sensitivities to IRVs

Example: Elmore Delay Calculation



Rs and Cs calculated from five independent geometrical parameters

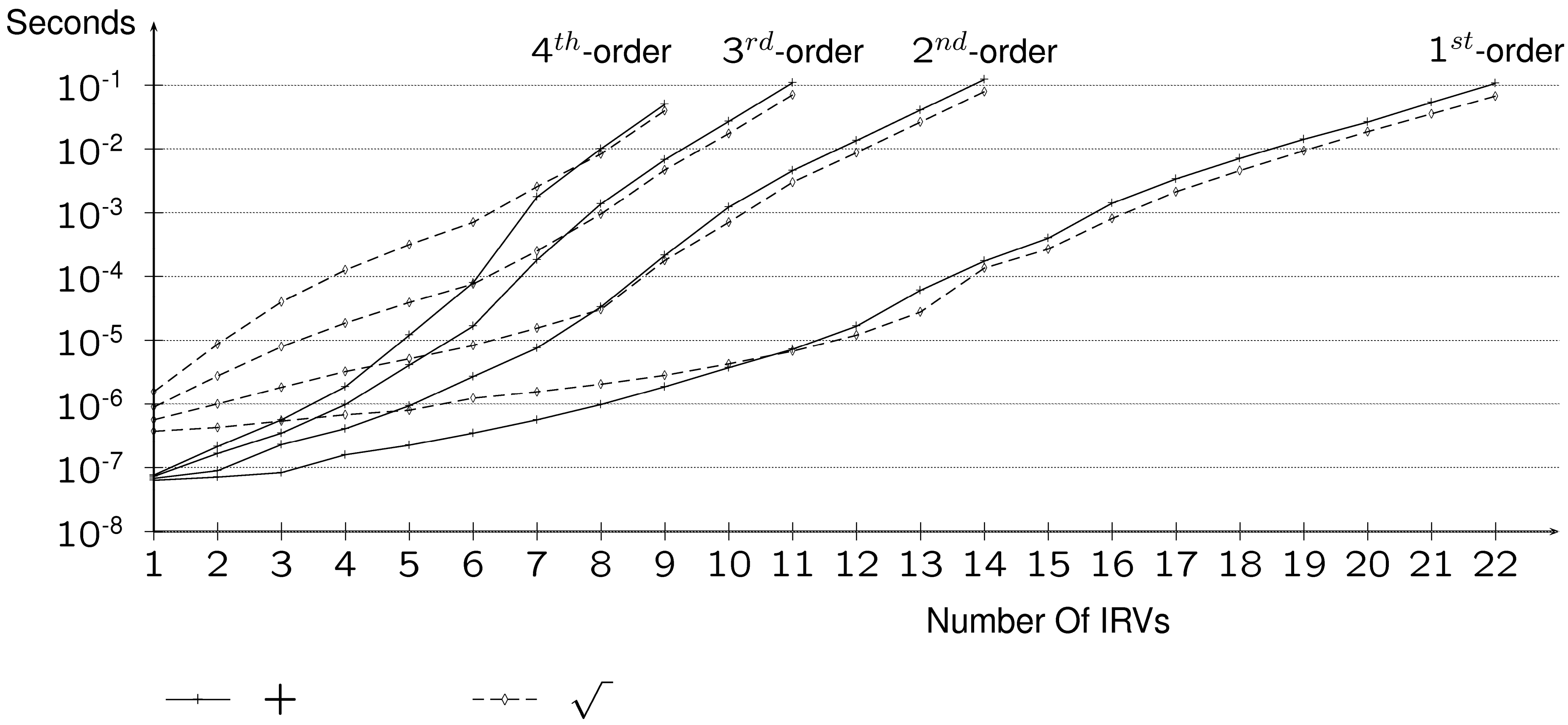
Standard deviation of 10 % is assumed

$$
\begin{aligned}
\tau = \; & (R_D + R_1)C_1 \\
& +(R_D + R_1 + R_2)C_2 \\
& +(R_D + R_1 + R_2 + R_3)C_3 \\
& +(R_D + R_1 + R_2 + R_3 + R_4)(C_4 + C_g) \\
& +(R_D + R_1)(C_5 + C_6 + C_g)
\end{aligned}
$$

Error and Runtimes of Elmore Delay Calculation

Order	μ_T	σ_T	Skewness	Kurtosis	Runtime
1	0.3 %	1.5 %	100 %	790 %	0.00 ms
2	0.0 %	1.2 %	7.1 %	99.6 %	0.01 ms
3	0.0 %	0.1 %	3.9 %	18.1 %	0.04 ms
4	0.0 %	0.1 %	0.4 %	4.4 %	0.13 ms
MC 10^4	0.2 %	0.4 %	11.5 %	24.4 %	6 ms
MC 10^9					454 s

Runtimes of Basic Operations



Conclusion

- Distribution Arithmetic allows to calculate with RVs as easy as with real values
- Correllations between RVs are considered
- RV moments can be determined directly
- Fast and accurate
- Not restricted to normal distributions
- Selectable tradeoff between runtime and accuracy by choosing DA order

Future Work

- Equation solver using DA variables
- DC and transient simulation with DA values
- Speed up by automatic neglect of non important coefficients

Thank you for your attention.