Gen-Adler: The Generalized Adler's Equation for Injection Locking Analysis in Oscillators

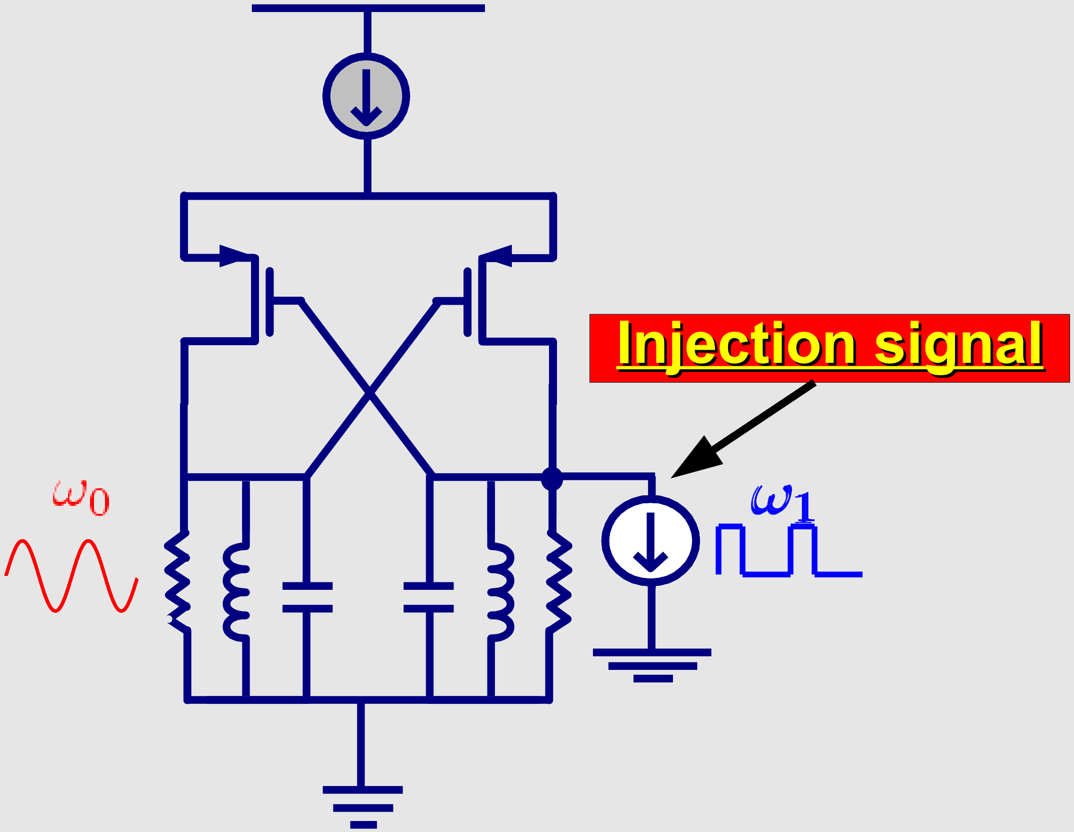
Prateek Bhansali, Jaijeet Roychowdhury

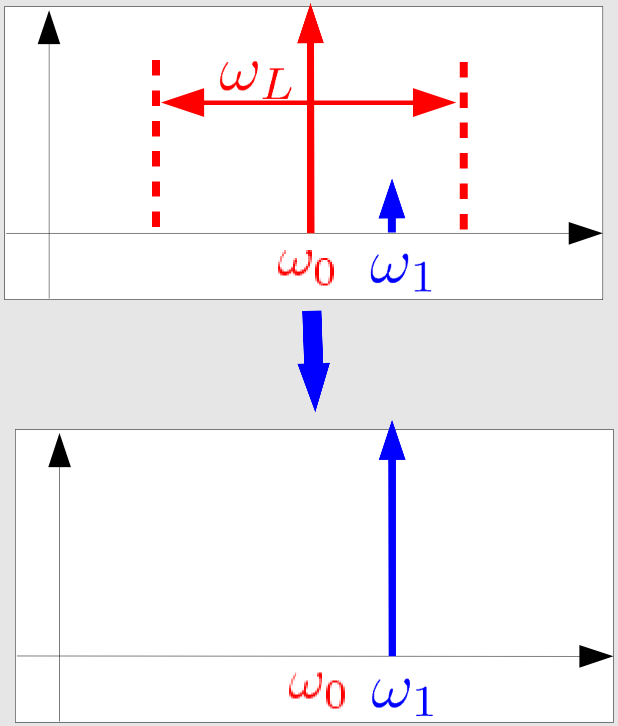
University of Minnesota, USA

Outline

- Introduction
  - Challenges involved
- Previous work
  - Adler's equation
- Gen-Adler injection locking equations
  - Ring oscillator
    - Sinusoidal, exponential, square injection signal
- Experimental results
- Conclusion

Introduction





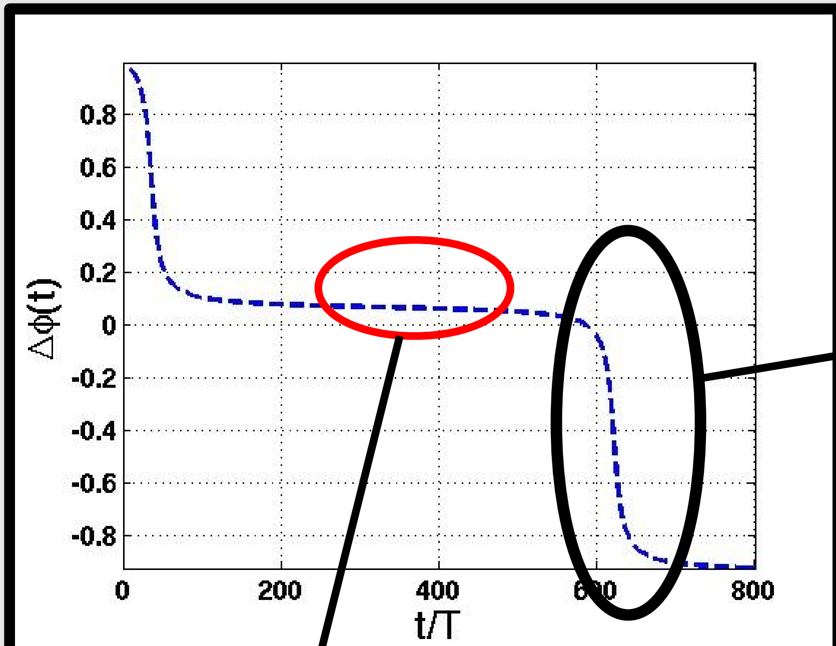
- Injection locking
  - **Frequency and phase are locked**
- Engineering Applications
  - variable phase shifts, frequency multiplication, low power frequency dividers, precision quadrature generation

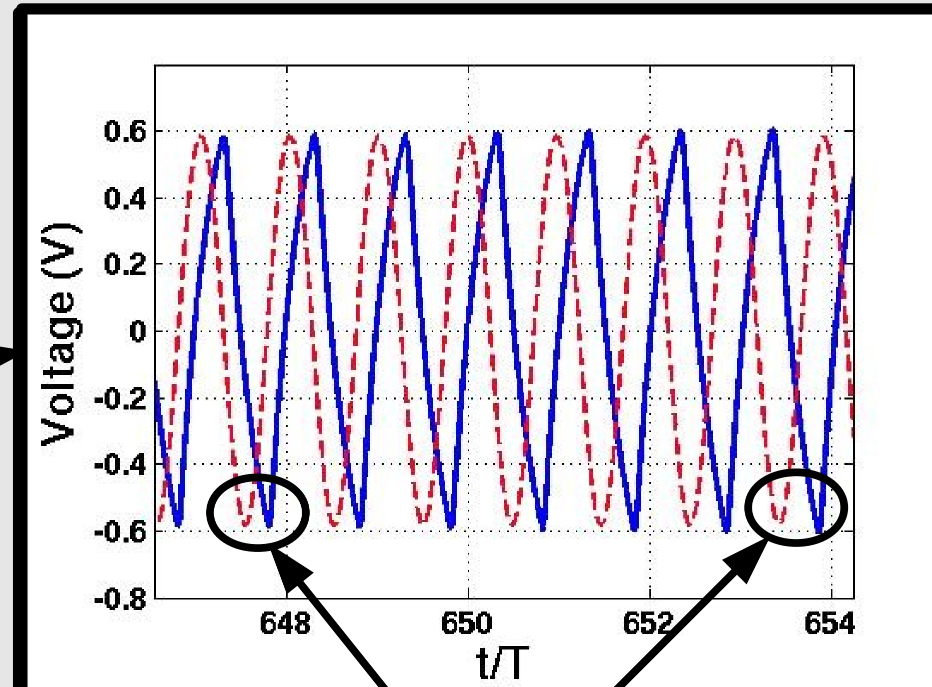
SPICE-level Simulation of Injection Locking

- **Inefficient and inaccurate**
  - Direct simulation of oscillators
    - → Extremely small time steps are required
    - → Accumulation of phase error
  - Difficult to extract phase and frequency information
  - Locking process can take several cycles
    - → Simulation for hundreds of cycles to conclusively declare oscillator locked or unlocked.
      - Distinction between lock and quasi-lock, occurs when injection frequency is just outside the locking range

SPICE-level Simulation of Injection Locking (Cont'd)

Oscillator Waveform —— Injection Signal - - -





Oscillator "appears" to be locked

Not Locked

- Several number of simulations required to determine the oscillator's locking range

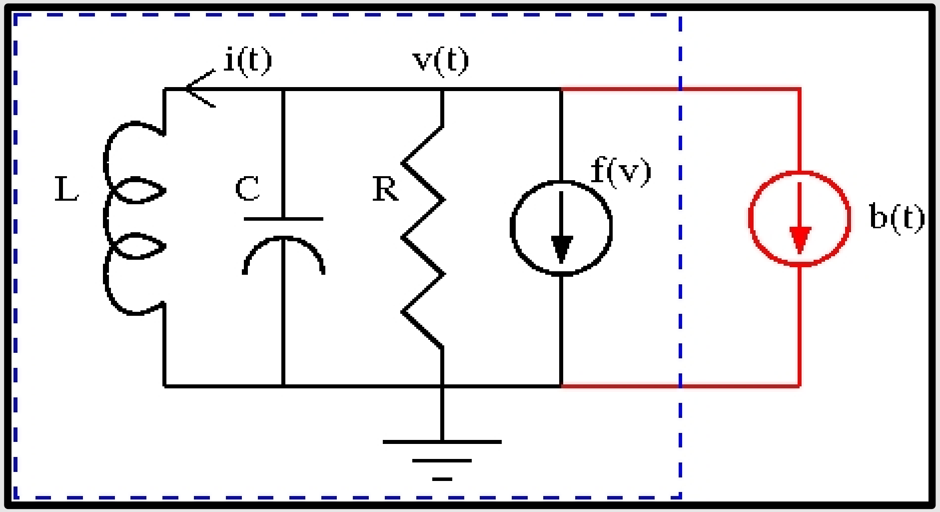
Alternative to SPICE-level simulation required

Previous Work on Injection Locking

1. Adler, R., "A Study of Locking Phenomena in Oscillators," Proceedings of the IRE , vol.34, no.6, pp. 351-357, June 1946
2. Razavi, B., "A study of injection locking and pulling in oscillators," Solid-State Circuits, IEEE Journal of , vol.39, no.9, pp. 1415-1424, Sept. 2004
3. Xiaolue Lai; Roychowdhury, J., "Capturing oscillator injection locking via nonlinear phase-domain macromodels," Microwave Theory and Techniques, IEEE Transactions on , vol.52, no.9, pp. 2251-2261, Sept. 2004
4. Xiaolue Lai; Roychowdhury, J., "Analytical equations for predicting injection locking in LC and ring oscillators," Custom Integrated Circuits Conference, 2005. Proceedings of the IEEE 2005 , vol., no., pp. 461-464, 18-21 Sept. 2005
5. Gourary, M.M.; Rusakov, S.G.; Ulyanov, S.L.; Zharov, M.M.; Mulvaney, B.J.; Gullapalli, K.K., "Injection locking conditions under small periodic excitations," Circuits and Systems, 2008. ISCAS 2008. IEEE International Symposium on , vol., no., pp.544-547, 18-21 May 2008
6. Gourary, M.M.; Rusakov, S.G.; Ulyanov, S.L.; Zharov, M.M.; Mulvaney, B.J.; Gullapalli, K.K., "Smoothed form of nonlinear phase macromodel for oscillators," Computer-Aided Design, 2008. ICCAD 2008. IEEE/ACM International Conference on , vol., no., pp.807-814, 10-13 Nov. 2008

Original Adler's Equation

Adler's Equation



$$\mathrm{v}(t) = A\cos(2\pi f_0 t)$$

$$b(t) = I_i \cos(2\pi f_1 t)$$

Adler's Equation

$$\frac{d\Delta\phi(t)}{dt} = \Delta f_0 - \frac{I_i}{I_R}\frac{f_0}{2Q}\sin(2\pi\Delta\phi(t))$$

$\Delta\phi(t)$	Phase difference
Δf_0	Frequency difference
Q	Quality Factor

Injection Locking Dynamics

$$\frac{d\Delta\phi(t)}{dt} = \Delta f_0 - \frac{I_i}{I_R}\frac{f_0}{2Q}\sin(\Delta\phi(t))$$

- Adler's equation provides quick insight into locking dynamics
  - Instantaneous phase difference, $\Delta\phi(t)$
  - Instantaneous frequency of oscillator, $f_{inst} = f_1 + \dfrac{d\Delta\phi(t)}{dt}$
- In steady state, when oscillator is injection locked

$$\Delta\phi(t) = \text{constant}$$

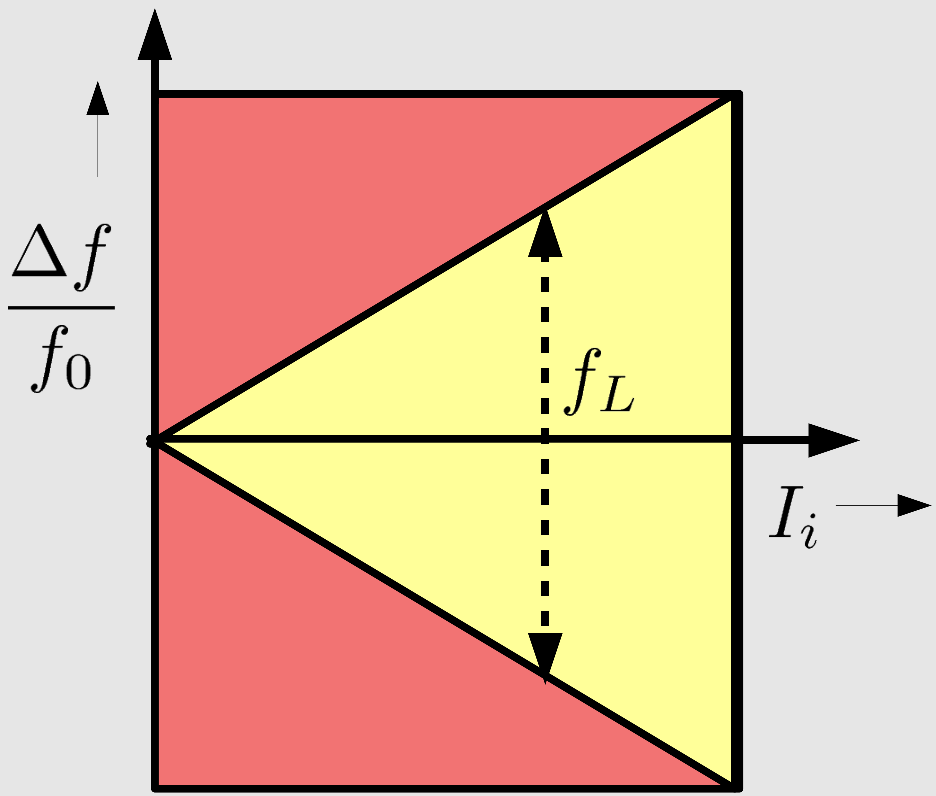
$$\Rightarrow \frac{d\Delta\phi(t)}{dt} = 0 \quad \Rightarrow \frac{\Delta f_0}{f_0} = \frac{I_i}{I_R}\frac{1}{2Q}\sin(\Delta\phi_0)$$

- Analytical equation relating locking range and injection amplitude

Locking Range

$$f_L = 2\left|\frac{\Delta f_0}{f_0}\right|_{max} = \frac{1}{Q}\frac{I_i}{I_R}$$

Locking Range



- Applicable only to **LC oscillator** (Q explicitly required) with **sinusoidal injection signal**

Review of Perturbation Projection Vector (PPV)

$$\frac{d\vec{x}}{dt} + \vec{f}(\vec{x}) = \vec{b}(t)$$

$\vec{x}$	Oscillator state variables
$\vec{f}$	Resistive components
$\vec{b}$	Perturbation to the oscillator
t	Time

$$\frac{d\vec{x}}{dt} + \vec{f}(\vec{x}) = 0 \rightarrow \vec{x}_{ss}(t) = [i_L(t), v_C(t)]^T$$

$$\frac{d\vec{x}}{dt} + \vec{f}(\vec{x}) = \vec{b}(t) \rightarrow \vec{x}_p(t)$$

$$\vec{x}_p(t) = \vec{x}_{ss}(t + \alpha(t))$$

Injection signal

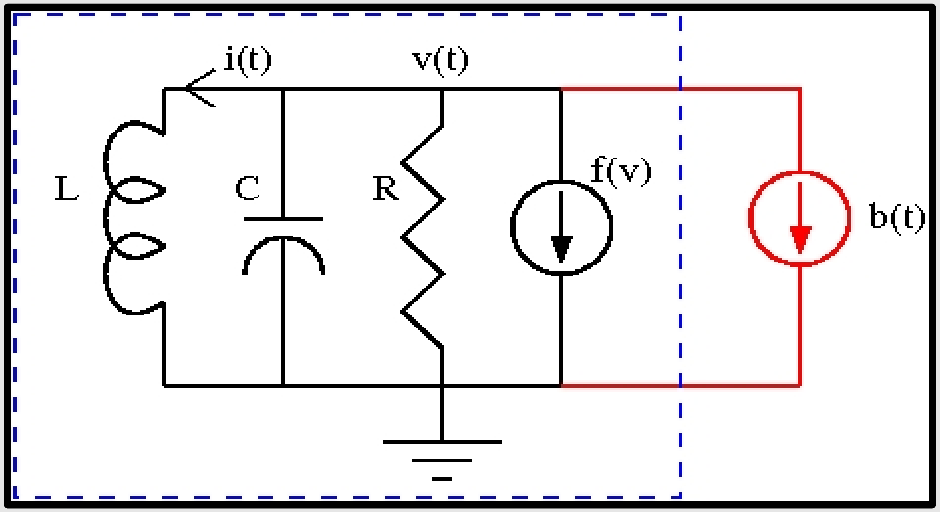
PPV equation

$$\dot{\alpha}(t) = \vec{v}_1^T(t + \alpha(t)) \cdot \vec{b}(t)$$

$$\phi(t) = f_0(t + \alpha(t))$$

Oscillator phase

PPV Equation for LC Oscillator



$$\mathrm{v}(t) = A\cos(2\pi f_0 t)$$

$$v_1^T(t) = -\sqrt{\frac{L}{C}}\frac{1}{A}\sin(2\pi f_0 t)$$

$$b(t) = I_i \cos(2\pi f_1 t)$$

PPV equation

$$\frac{d\alpha(t)}{dt} = \vec{v}_1^T\,(t+\alpha(t))\;\cdot\;\vec{b}(t)$$

$$\frac{d\alpha(t)}{dt} = -\sqrt{\frac{L}{C}}\frac{1}{A}\sin(2\pi f_0(t+\alpha(t)) \cdot I_i\cos(2\pi f_1 t)$$

$$\phi_1 = f_1 t$$

$$\phi = f_0(t+\alpha(t))$$

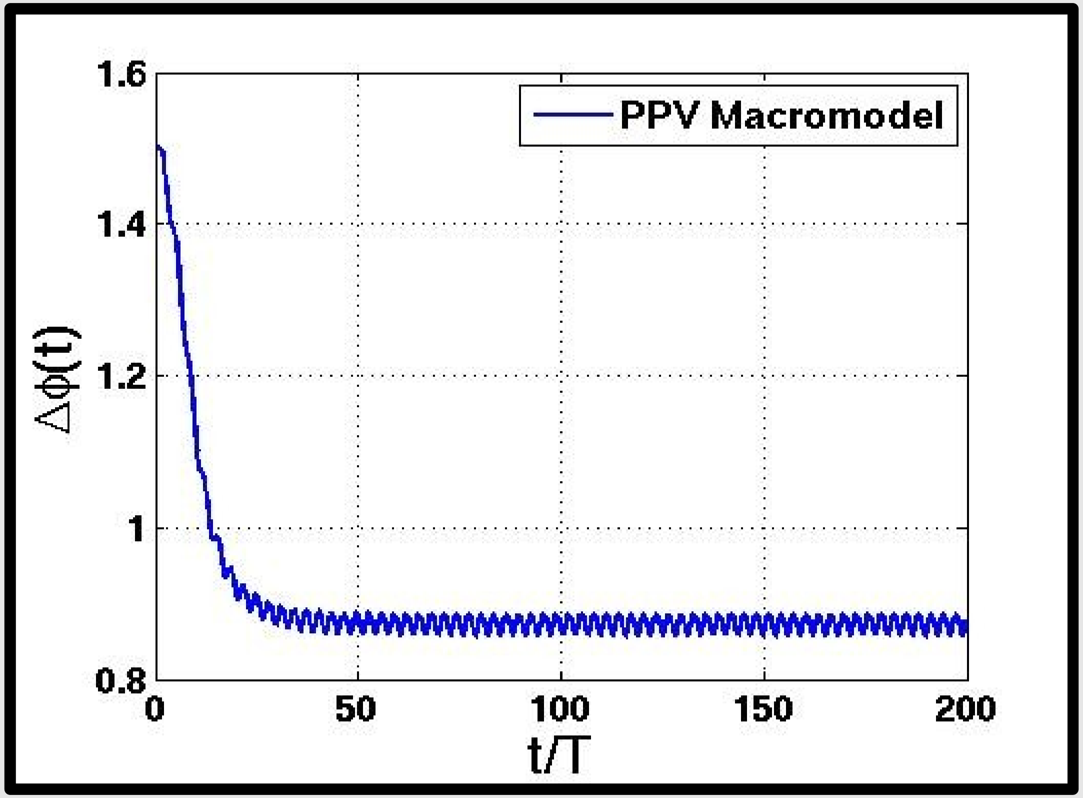
$$\Delta\phi(t) = \phi(t) - \phi_1(t)$$

$$\alpha(t) = \frac{\Delta\phi}{f_0} + \frac{f_1 - f_0}{f_0}t$$

$$\frac{d\alpha(t)}{dt} = \frac{1}{f_0}\frac{d\Delta\phi}{dt} + \frac{f_1 - f_0}{f_0}$$

PPV Equation and Phase Difference

$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) - \frac{I_i}{I_R}\frac{f_0}{2Q}\left[\sin(2\pi\Delta\phi) + \sin(2\pi\Delta\phi + 2\pi f_1 t)\right]$$



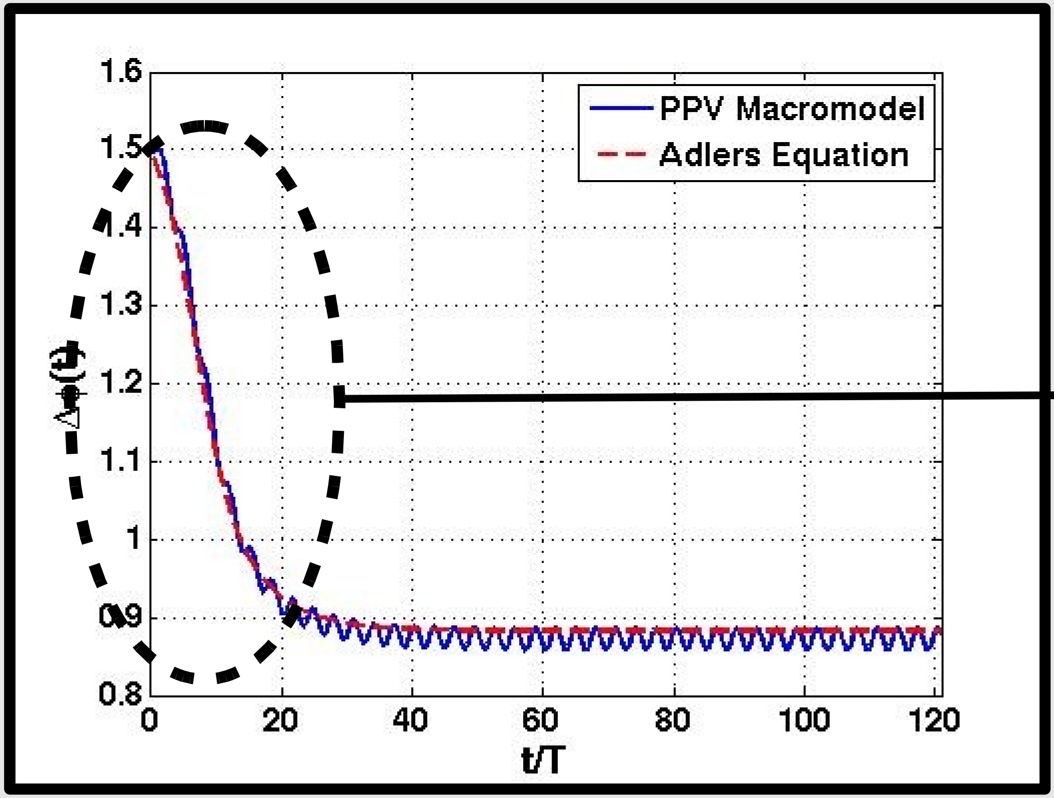
PPV Equation and Adler's Equation

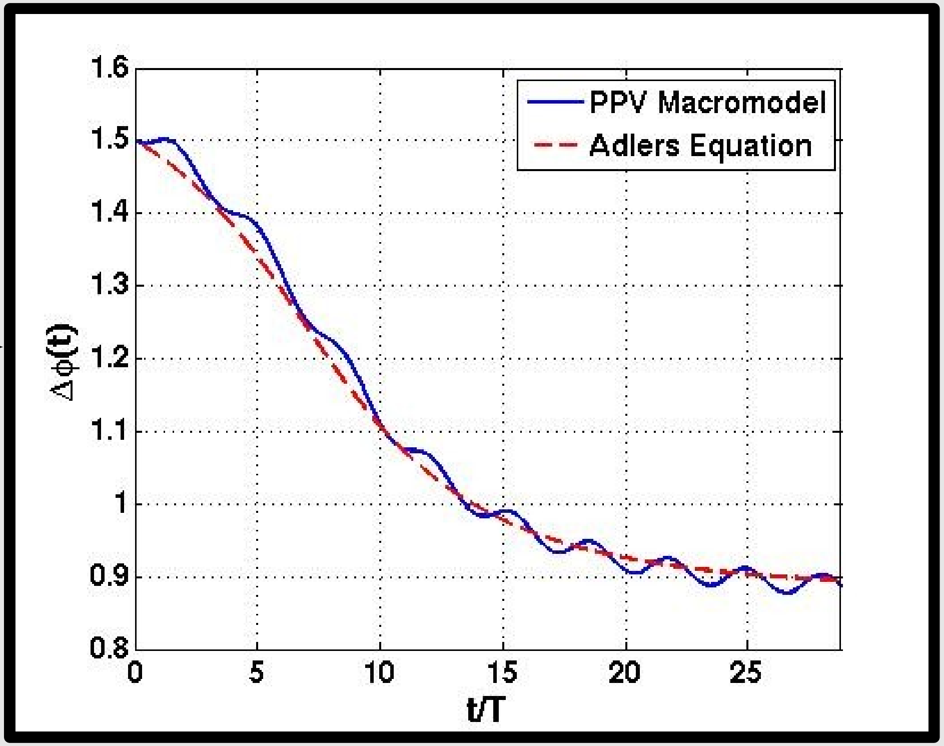
$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) - \frac{I_i}{I_R}\frac{f_0}{2Q}\left[\sin(2\pi\Delta\phi)\right]$$

"fast" varying

$$\phi_1 = f_1 t$$

$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) - \frac{I_i}{I_R}\frac{f_0}{2Q}\left[\sin(2\pi\Delta\phi) + \sin(2\pi\Delta\phi + 2\pi f_1 t)\right]$$





Adler's Equation from PPV Equation

PPV equation

$$\phi_1 = f_1 t$$

$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) - \frac{I_i}{I_R}\frac{f_0}{2Q}\left[\sin(2\pi\Delta\phi) + \sin(2\pi\Delta\phi + 2\pi f_1 t)\right]$$

$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) - f_0 g(\Delta\phi, \phi_1)$$

Average over "fast" varying variable $\phi_1 = f_1 t$

$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) - f_0 g(\Delta\phi)$$

$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) - f_0 \frac{I_i}{I_R}\frac{1}{2Q}\left[\sin(2\pi\Delta\phi)\right]$$

Adler's equation

Gen-Adler: Generalized Adler's Equation

Generalized Adler's Equation and PPV Equation

PPV equation

$$\frac{d\alpha(t)}{dt} = \vec{v}_1^T(t+\alpha(t)) \cdot \vec{b}(t) \qquad (1)$$

Step 1: $\vec{v}_1^T(t) = \vec{\chi}(f_0 t)$

$$\frac{d\alpha(t)}{dt} = \vec{\chi}(f_0(t+\alpha(t))) \cdot \vec{b}(f_1 t) \qquad (2)$$

Step 2: $\Delta\phi(t) = \phi(t) - \phi_1(t); \phi(t) = f_0(t+\alpha(t)); \phi_1(t) = f_1 t$

$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) + f_0\vec{\chi}(\Delta\phi(t) + \phi_1(t)) \cdot \vec{b}(\phi_1(t))$$

Modified phase equation

Generalized Adler's Equation

$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) + f_0 \underline{\vec{\chi}(\Delta\phi(t) + \phi_1(t)) \cdot \vec{b}(\phi_1(t))}$$

Step 3: Average over the "fast" varying variable

$$g(\Delta\phi(t)) = \frac{1}{T_1}\int_0^{T_1} \chi(\Delta\phi(t) + \phi_1(t)) \cdot b(\phi_1(t))\, d\phi_1(t)$$

where, $T_1 = \phi_1(\frac{1}{f_1})$

$$\boxed{\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) + f_0 g(\Delta\phi(t))}$$

$$\boxed{\left(\frac{d\Delta\phi(t)}{dt}\right)_{max} = -(f_1 - f_0) + f_L/2} \ll \boxed{\frac{d\phi_1(t)}{dt} = f_1}$$

slow **fast**

Generalized Adler's Equation Contd. ...

$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) + f_0 g(\Delta\phi(t))$$

- Same form as of original Adler's equation

$$\frac{d\Delta\phi(t)}{dt} = \Delta f_0 - f_0 \frac{I_i}{I_R} \frac{1}{2Q} \sin(\Delta\phi(t))$$

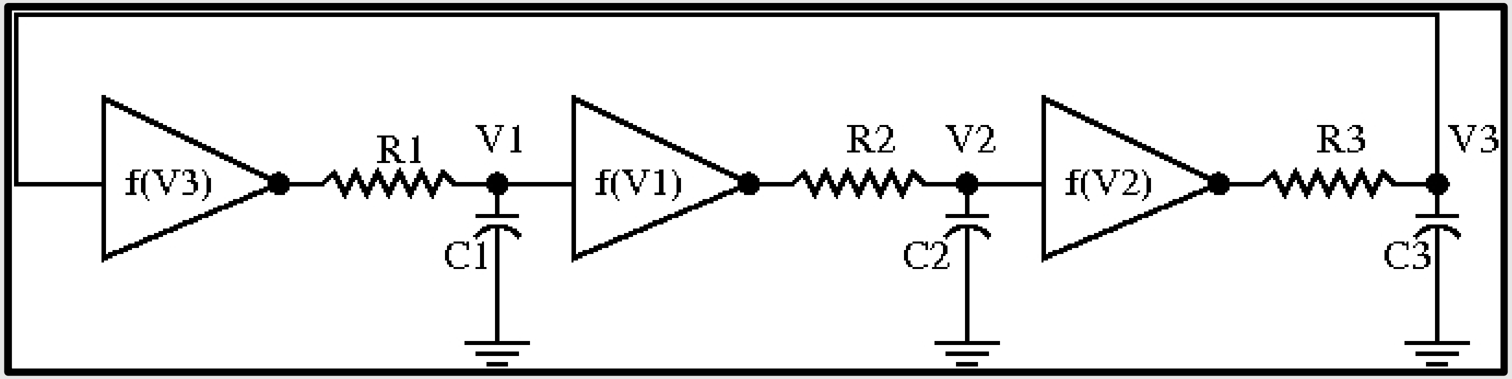
- Applicable for analysis of any oscillator unlike original Adler's Equation
- Any type of periodic injection signal: exponential, sinusoidal, square …
- Obtained by averaging accurate PPV equation, but has Adler like simplicity

$$g(\Delta\phi(t)) = \frac{1}{T_1} \int_0^{T_1} \chi(\Delta\phi(t) + \phi_1(t)) \cdot b(\phi_1(t))\, d\phi_1(t)$$

Analytical formulation

Analytical Formulation of Injection Locking Dynamics in Ring Oscillator

Injection Locking in Ring Oscillator

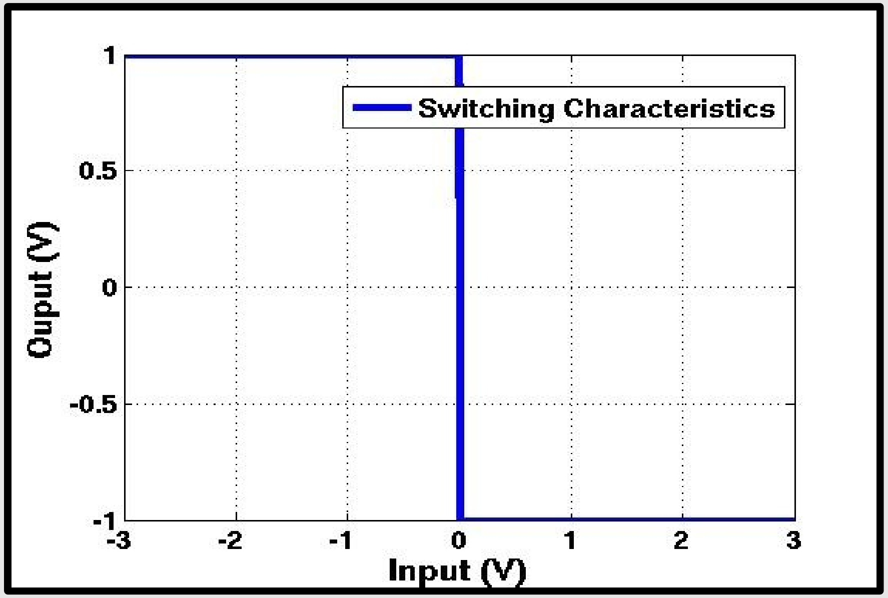


$$\frac{dv_1}{dt} = -\frac{v_1(t)}{R_1C_1} + \frac{\tanh(Gm_3v_3(t))}{R_1C_1}$$
$$\frac{dv_2}{dt} = -\frac{v_2(t)}{R_2C_2} + \frac{\tanh(Gm_1v_1(t))}{R_2C_2}$$
$$\frac{dv_3}{dt} = -\frac{v_3(t)}{R_3C_3} + \frac{\tanh(Gm_2v_2(t))}{R_3C_3}$$

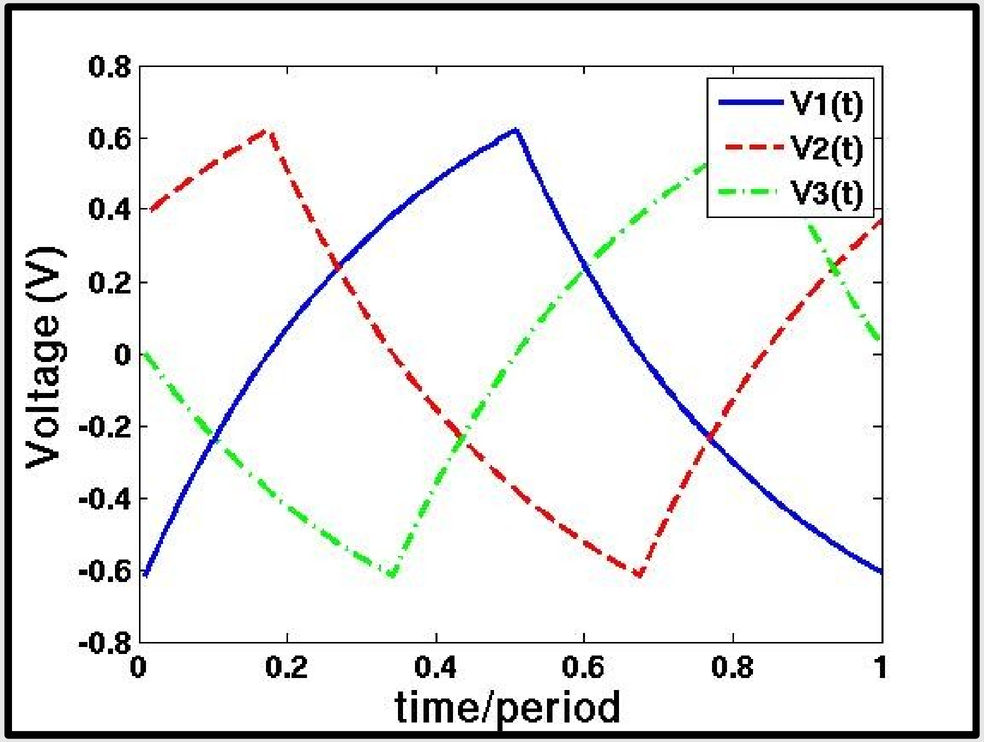
Three stage ring oscillator DEs

$$G_m \implies \infty$$

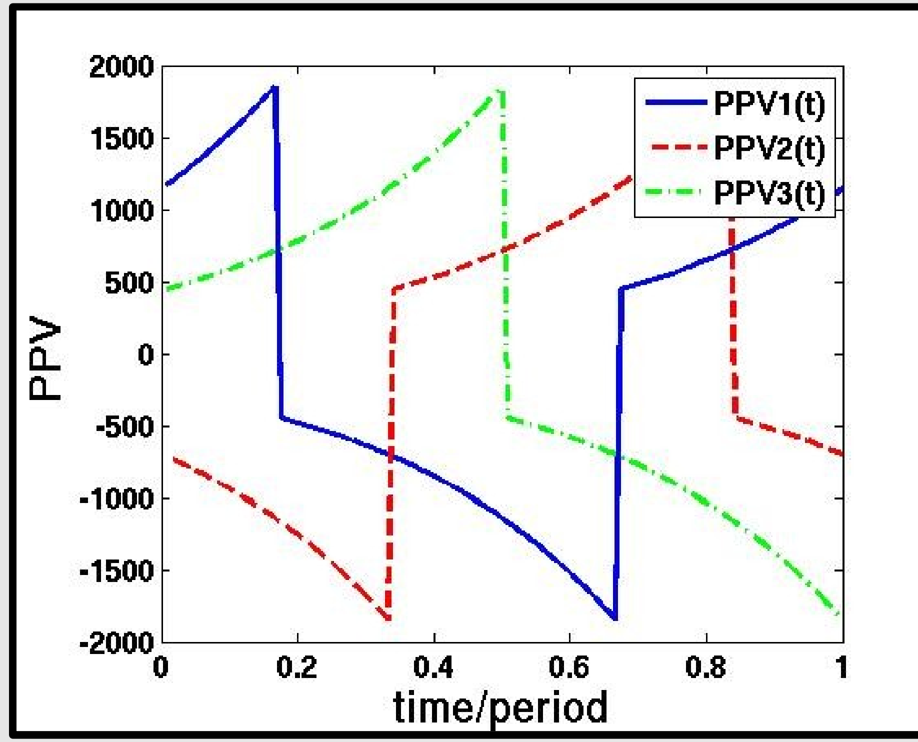
Ideal switching characteristics



Ring Oscillator's PPV



Steady State Waveforms

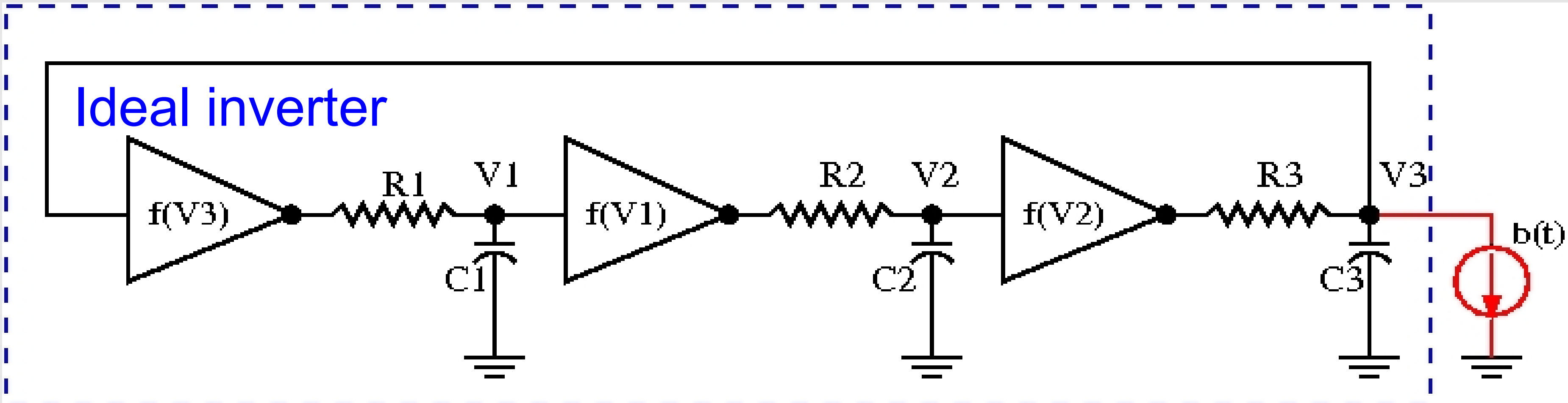


PPV Waveforms

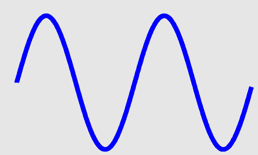
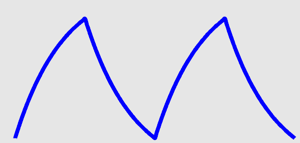
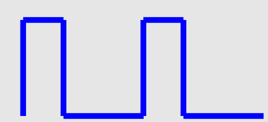
$$v_1^T(t) = \begin{cases} \frac{1}{\sqrt{5}} \frac{R}{A} e^{\frac{t}{\tau}} & \text{if } 0 \le t < \frac{T}{2} \\ \frac{R}{A}\left(\frac{2}{\sqrt{5}} - 1\right) e^{\frac{t}{\tau}} & \text{if } \frac{T}{2} \le t < T \end{cases}$$

PPV at node V3

Gen-Adler for Ring Oscillator



$$g(\Delta\phi(t)) = \frac{1}{T_1}\int_0^{T_1} \chi(\Delta\phi(t) + \phi_1(t)) \cdot b(\phi_1(t))\, d\phi_1(t)$$

- Sinusoidal injection signal 
- Exponential injection signal 
- Square injection signal (with any duty cycle) 

Analytical Injection Locking Dynamic Equations

$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) + f_0 g(\Delta\phi(t))$$

Sinusoidal Injection to a ring oscillator

$$g(\Delta\phi(t)) = \frac{1}{\sqrt{4\pi^2 + K_0^2}} \frac{RI_i}{A} \sin(2\pi\Delta\phi(t) + \zeta) \times \left[K_1(e^{K_0/2} + 1) - K_2(e^{K_0} + e^{K_0/2})\right]$$

where, $\sin(\zeta) = \dfrac{2\pi}{\sqrt{4\pi^2 + K_0^2}}$

$$K_0 = 2.887, \quad K_1 = \frac{1}{\sqrt{5}}, \quad K_2 = \left(\frac{2}{\sqrt{5}} - 1\right)$$

Injection Locking Range

- In steady state, when oscillator is injection locked

$$\Delta\phi(t) = \Delta\phi_0(\text{constant}) \quad \Rightarrow \quad \frac{d\Delta\phi(t)}{dt} = 0$$

$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) + f_0 g(\Delta\phi(t))$$

$$0 = -(f_1 - f_0) + f_0 g(\Delta\phi_0) \quad \Rightarrow \quad \frac{\Delta f_0}{f_0} = g(\Delta\phi_0)$$

$$\Delta f_{0_{max}} = f_0 \left[g(\Delta\phi_0(t))\right]_{max}$$

$$|\Delta f_0|_{max} = \frac{f_0}{\sqrt{4\pi^2 + K_0^2}} \frac{RI_i}{A} \left[K_1(e^{K_0/2} + 1) - K_2(e^{K_0} + e^{K_0/2})\right] = 0.6773 f_0 \frac{RI_i}{A}$$

$$f_L = 2\Delta f_{0_{max}}$$

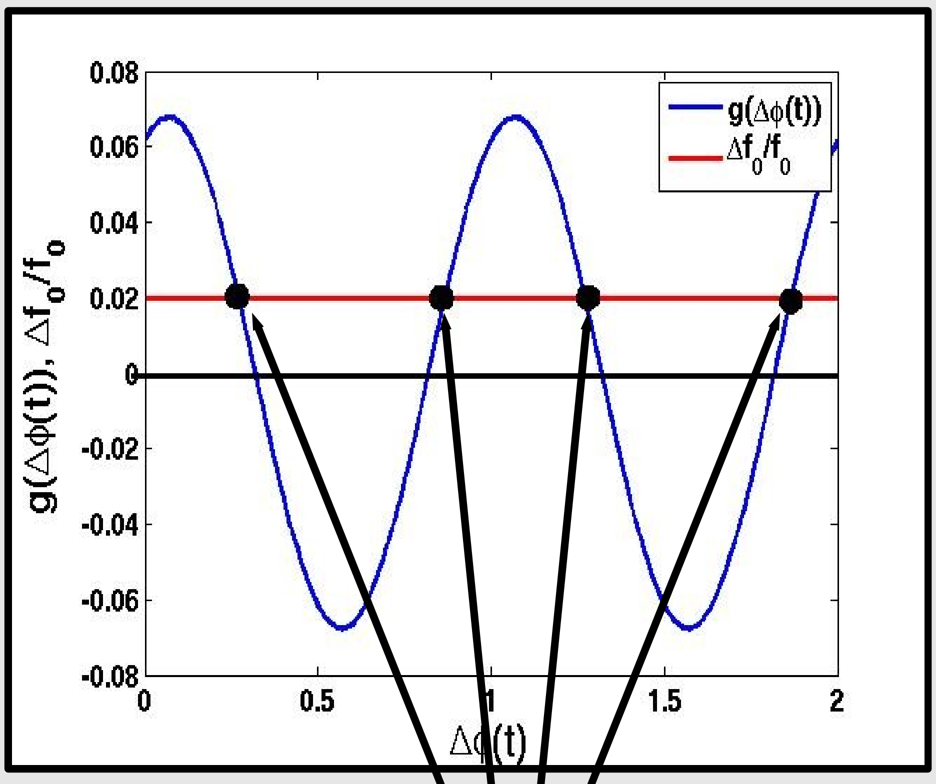
Lock Range

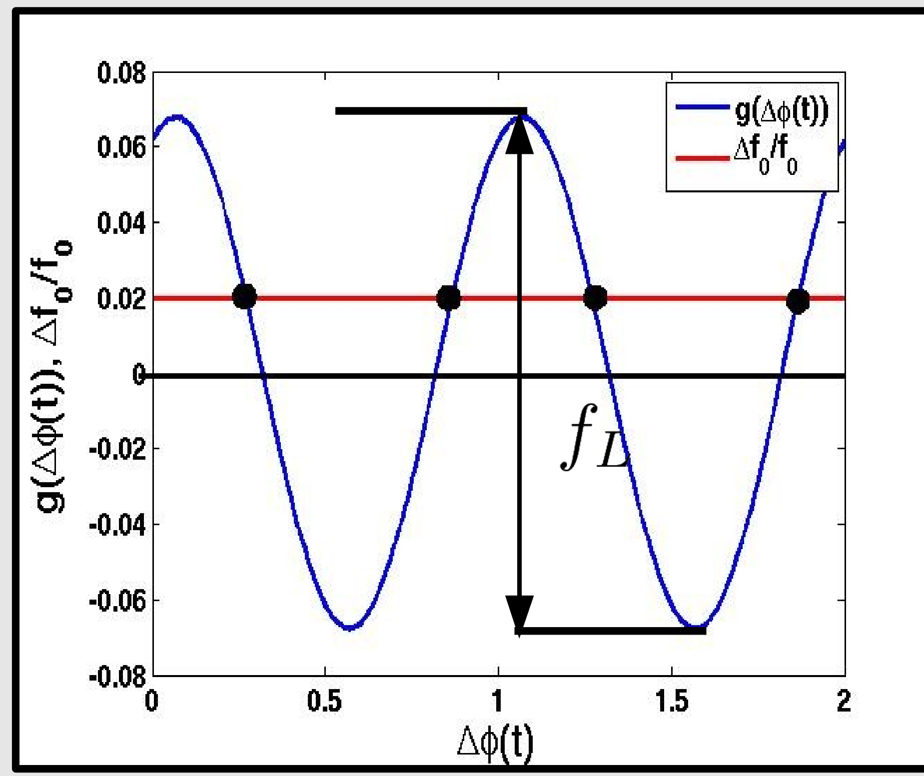
Graphical Injection Locking Analysis

$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) + f_0 g(\Delta\phi(t))$$

$$0 = -(f_1 - f_0) + f_0 g(\Delta\phi_0)$$

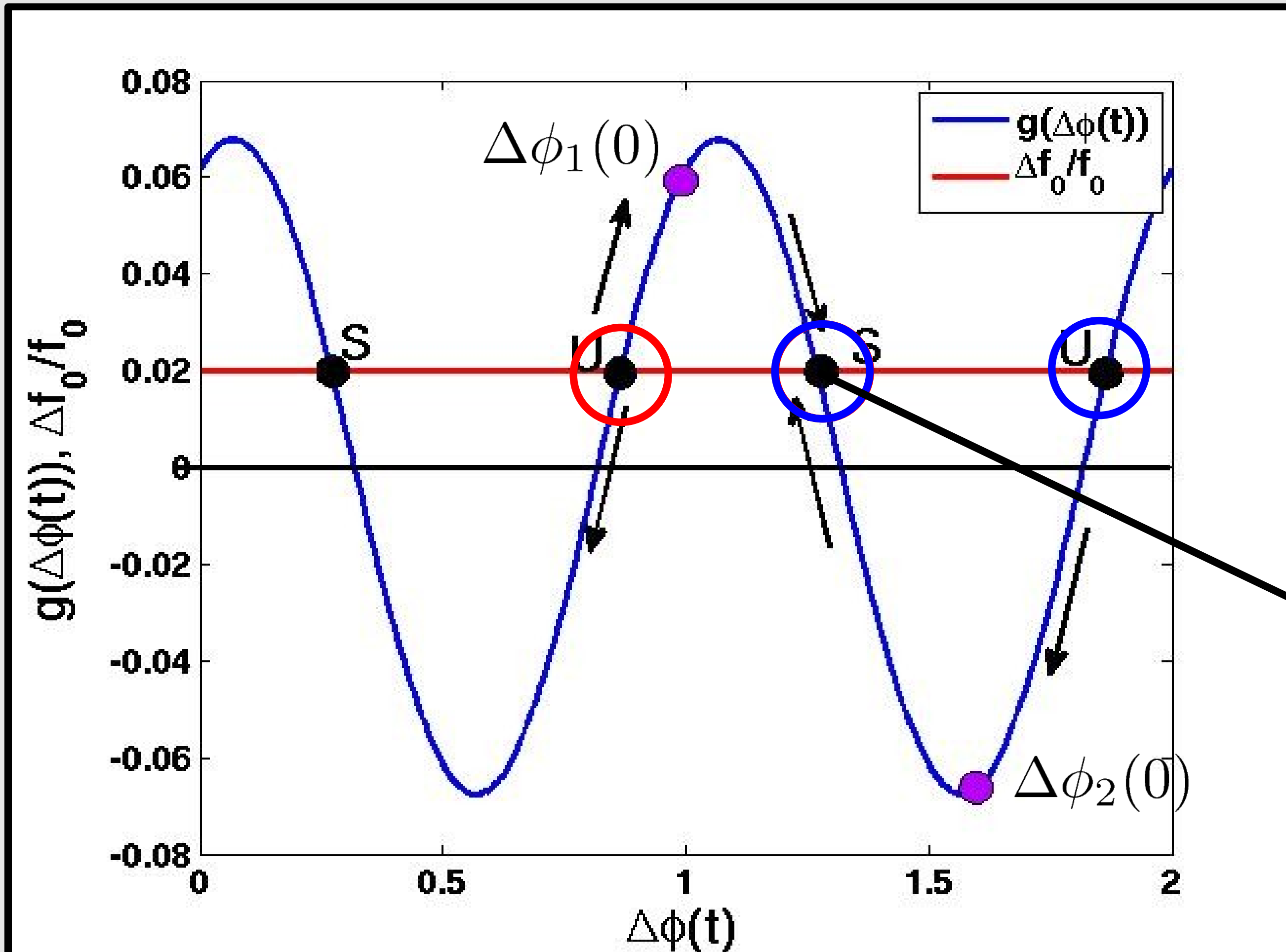
$$\frac{\Delta f_0}{f_0} = g(\Delta\phi_0)$$





Steady state phase

Graphical Injection Locking Analysis



$$\left(\frac{d\Delta\phi(t)}{dt}\right)_{\Delta\phi_1(0)} > 0$$

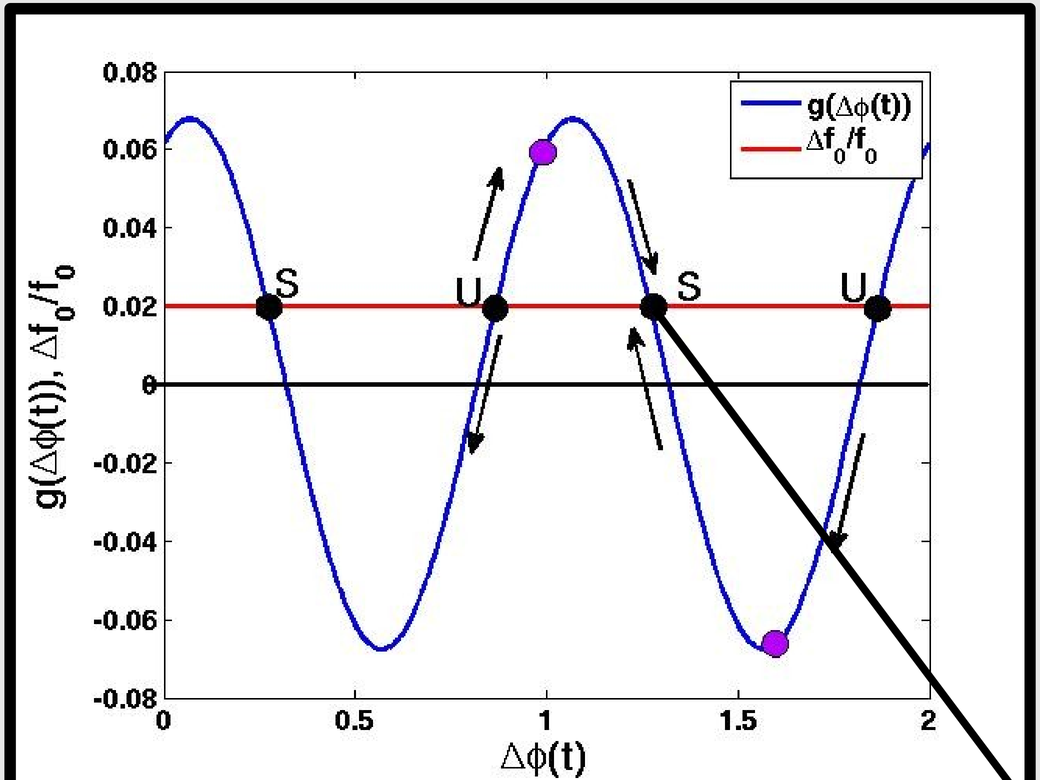
$$\Delta\phi_{ss} = 1.295$$

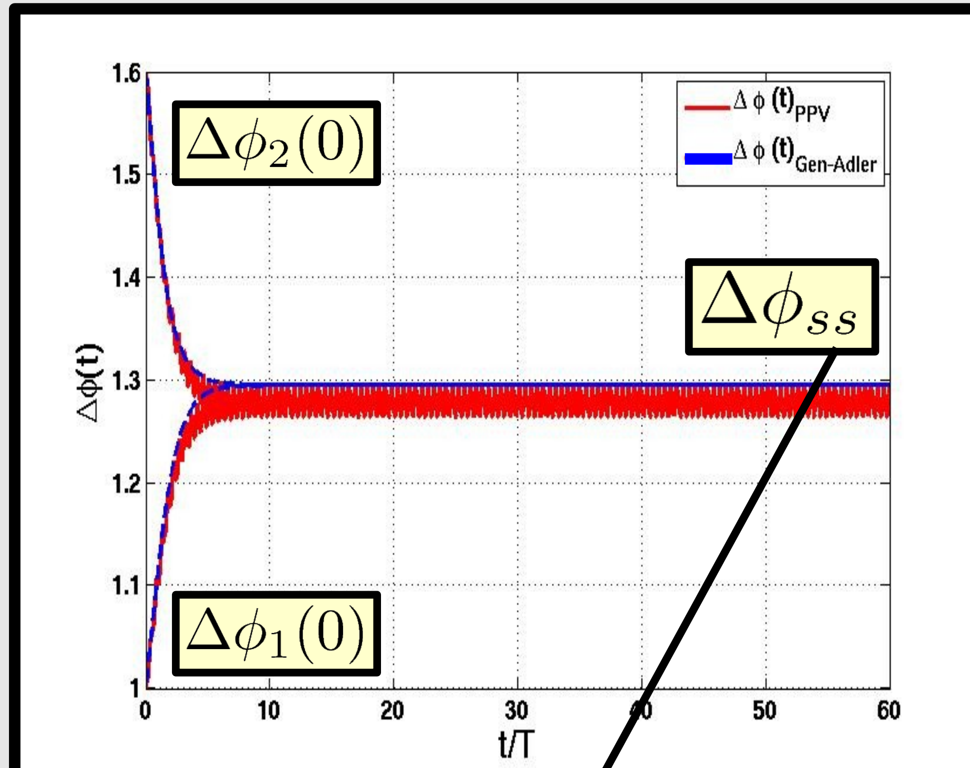
$$\left(\frac{d\Delta\phi(t)}{dt}\right)_{\Delta\phi_2(0)} < 0$$

$$\frac{d\Delta\phi(t)}{dt} = f_0(g(\Delta\phi(t)) - \frac{\Delta f_0}{f_0})$$

- **Unstable and stable steady state phase**

Graphical Injection Locking Analysis





$$\Delta\phi = 1.295$$

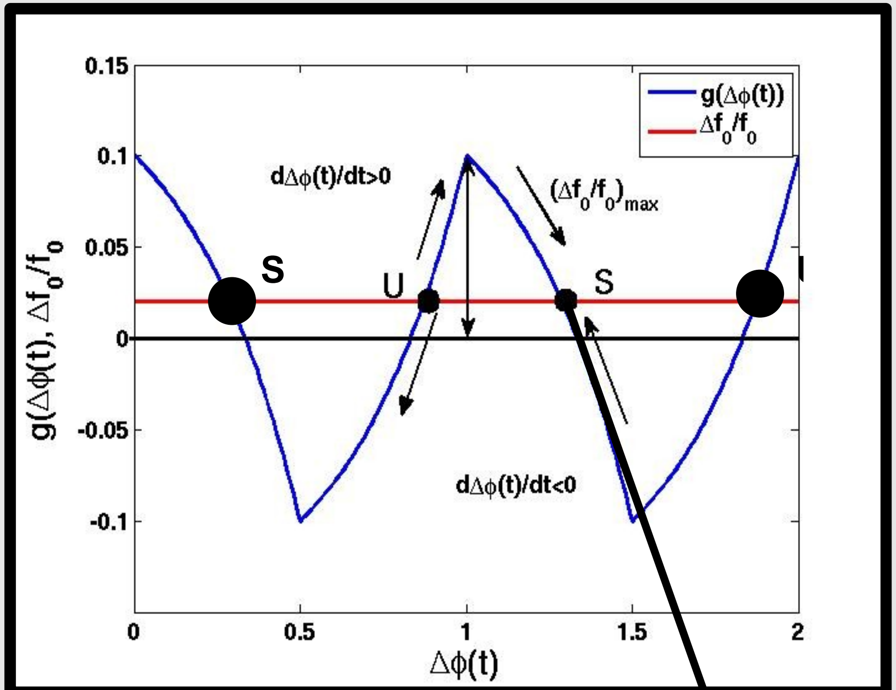
Square Wave Injection Signal

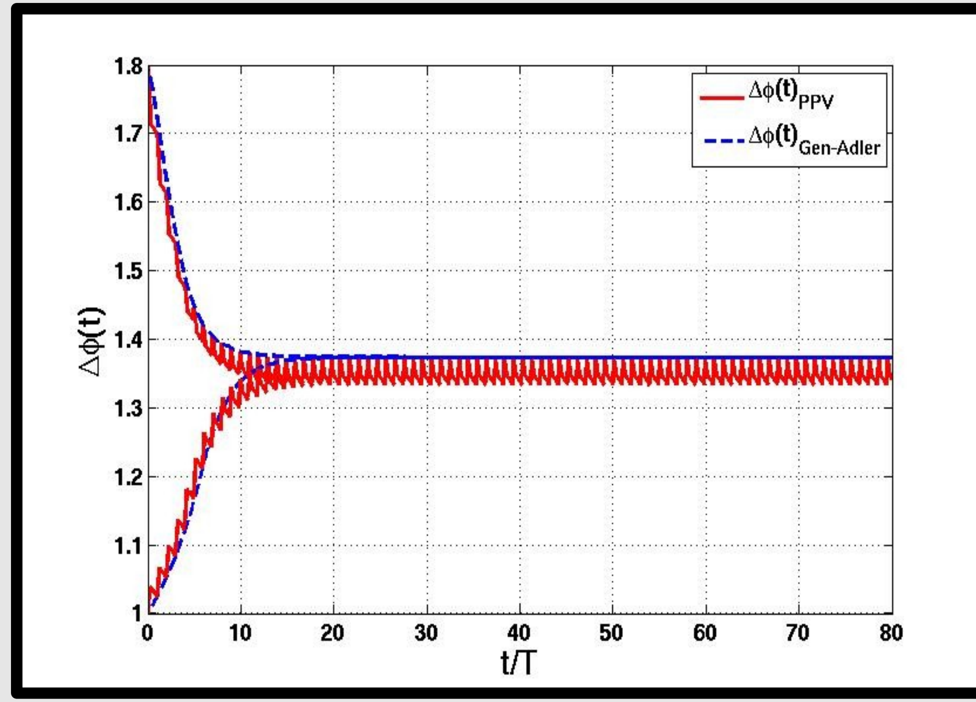
$$g(\Delta\phi(t)) = \begin{cases} \frac{RI_i}{A}\frac{K_1}{K_0}\left[e^{K_0\Delta\phi(t)}(e^{\eta K_0}-1)\right] \\ \qquad \text{if } 0 \le \Delta\phi(t) < \frac{1}{2}-\eta \\ \\ \frac{RI_i}{A}\frac{1}{K0}\left[(K_1-K_2)e^{K_0/2}+e^{K_0\Delta\phi(t)}(K_2e^{\eta K_0}-K_1)\right] \\ \qquad \text{if } \frac{1}{2}-\eta \le \Delta\phi(t) < \frac{1}{2} \\ \\ \frac{RI_i}{A}\frac{K2}{K0}\left[e^{K_0\Delta\phi(t)}(e^{\eta K_0}-1)\right] \\ \qquad \text{if } \frac{1}{2} \le \Delta\phi(t) < 1-\eta \\ \\ \frac{RI_i}{A}\frac{1}{K0}\left[(K_1-K_2)e^{K_0/2}+e^{K_0\Delta\phi(t)}(K_2-K_1e^{(\eta-1)K_0})\right] \\ \qquad \text{if } 1-\eta \le \Delta\phi(t) < 1 \end{cases}$$

$0 \le \eta < 0.5 \quad \eta =$ duty cycle of square wave

$$\frac{d\Delta\phi(t)}{dt} = -(f_1-f_0)+f_0g(\Delta\phi(t))$$

Square Wave Injection Signal



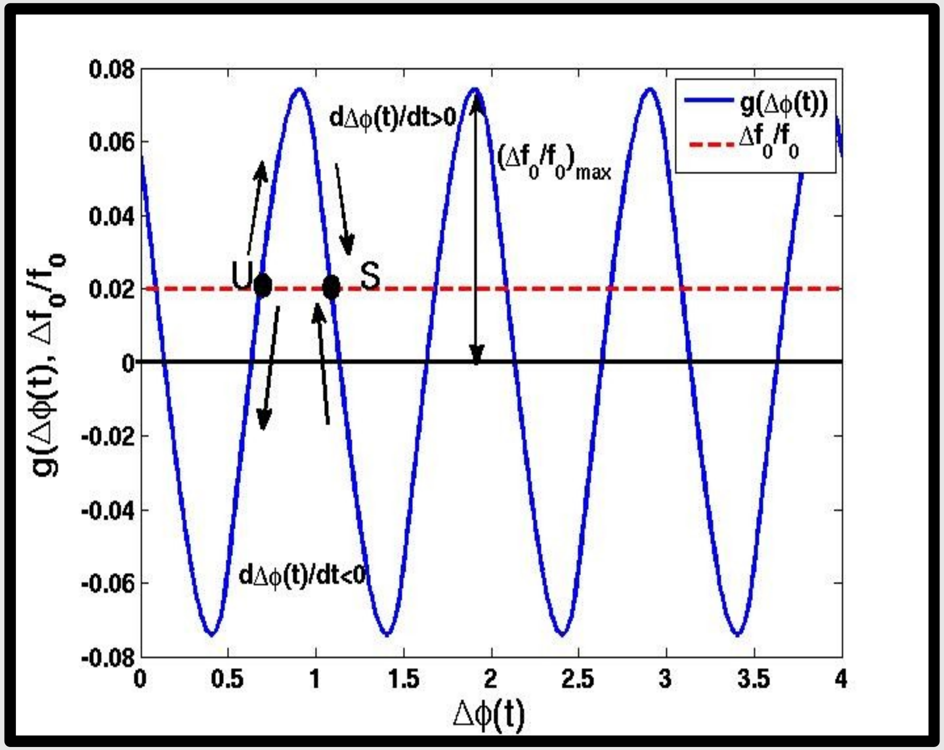


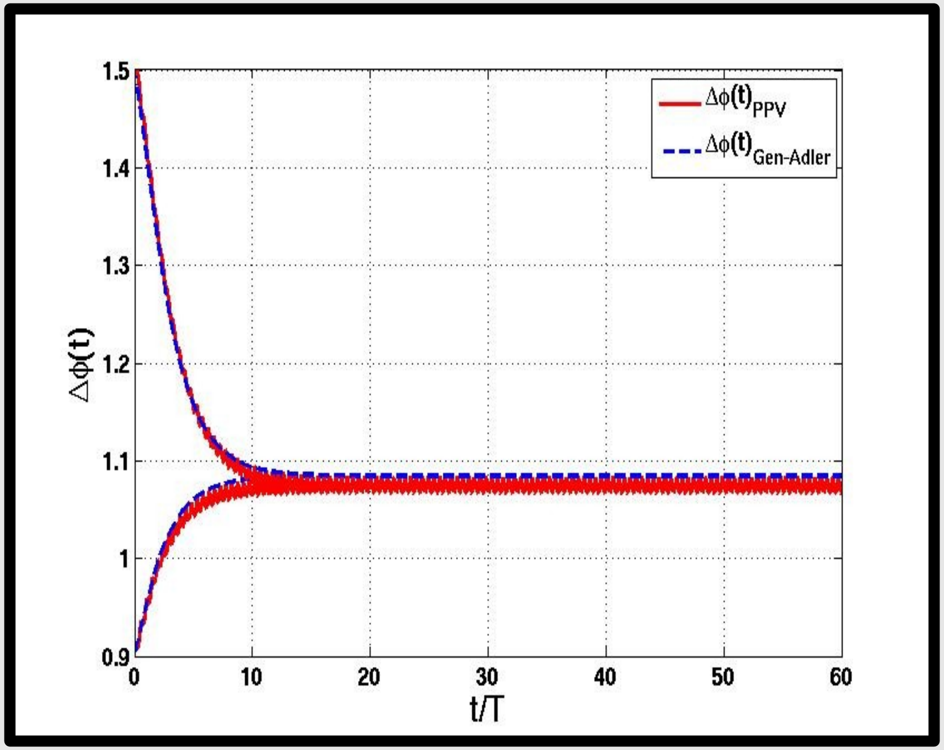
$$\Delta\phi = 1.3735$$

Lock Range

$$f_L = 2f_0 \frac{RI_i}{A}\frac{K_1}{K_0}\left[e^{K_0/2}(1 - e^{-\eta K_0})\right]$$

Exponential Injection Signal

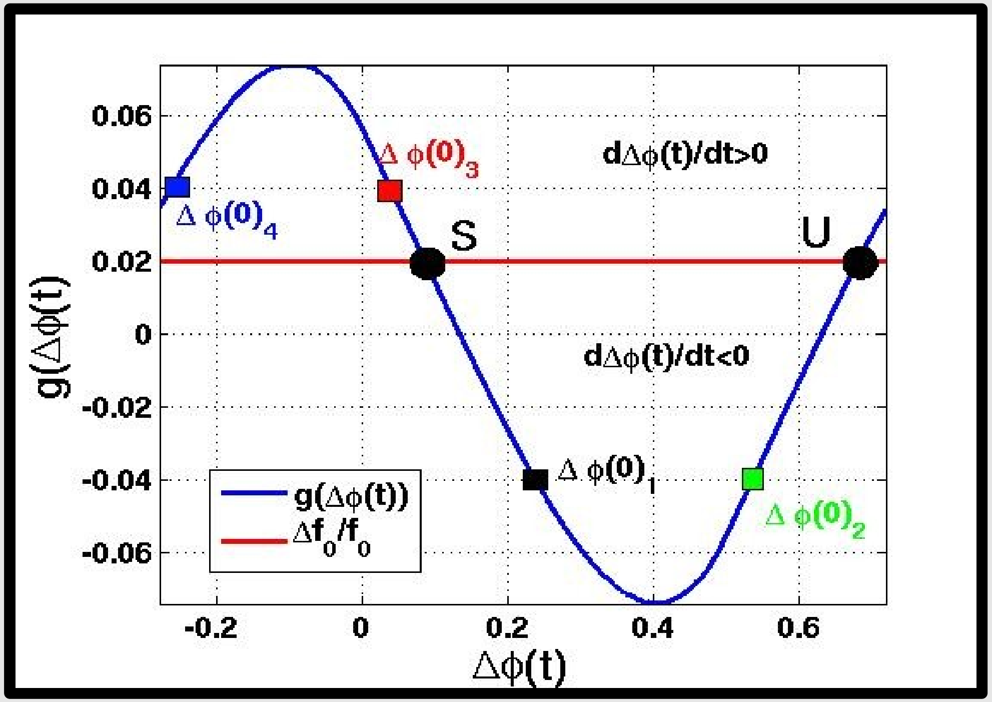


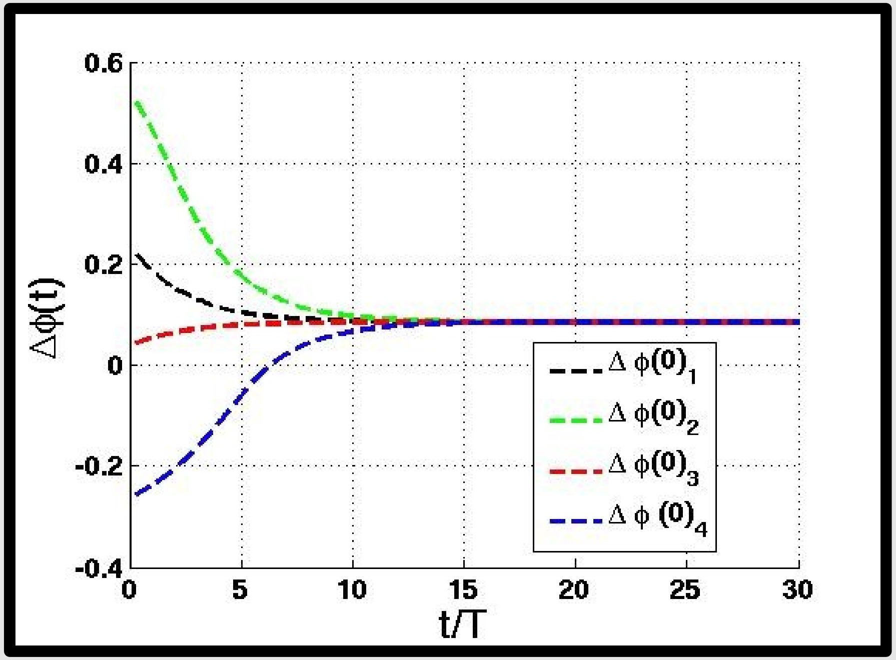


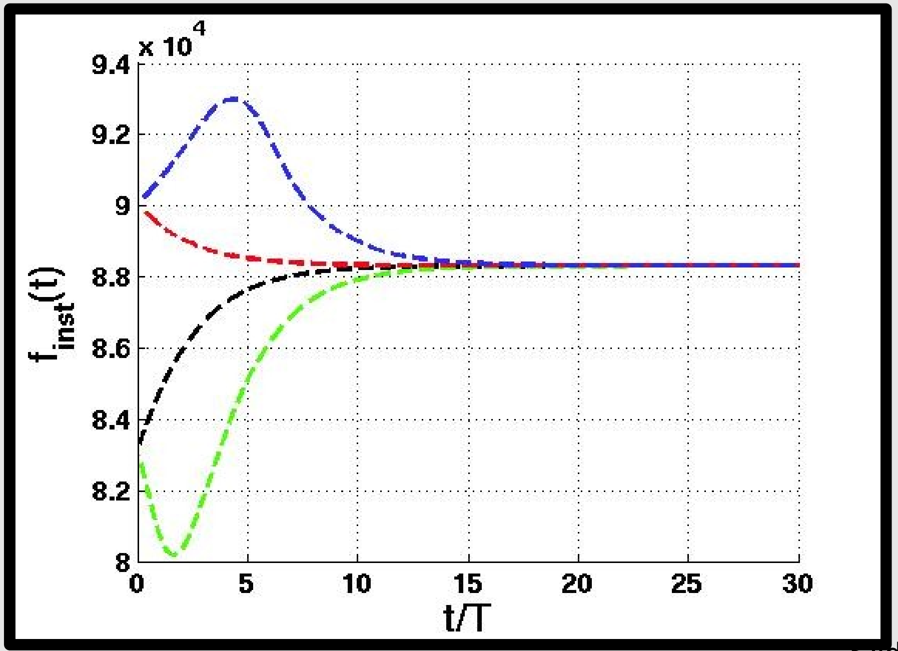
Lock Range

$$f_L = 0.744 f_0 \frac{RI_i}{A}$$

Instantaneous Phase and Frequency





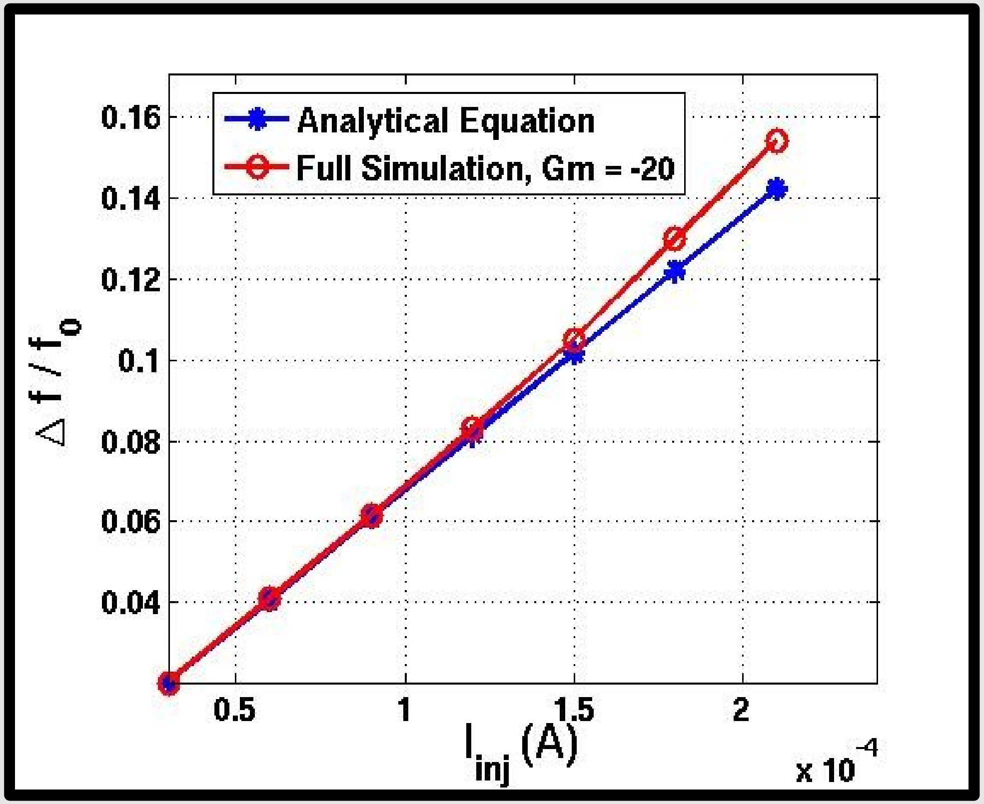


$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) + f_0 g(\Delta\phi(t))$$

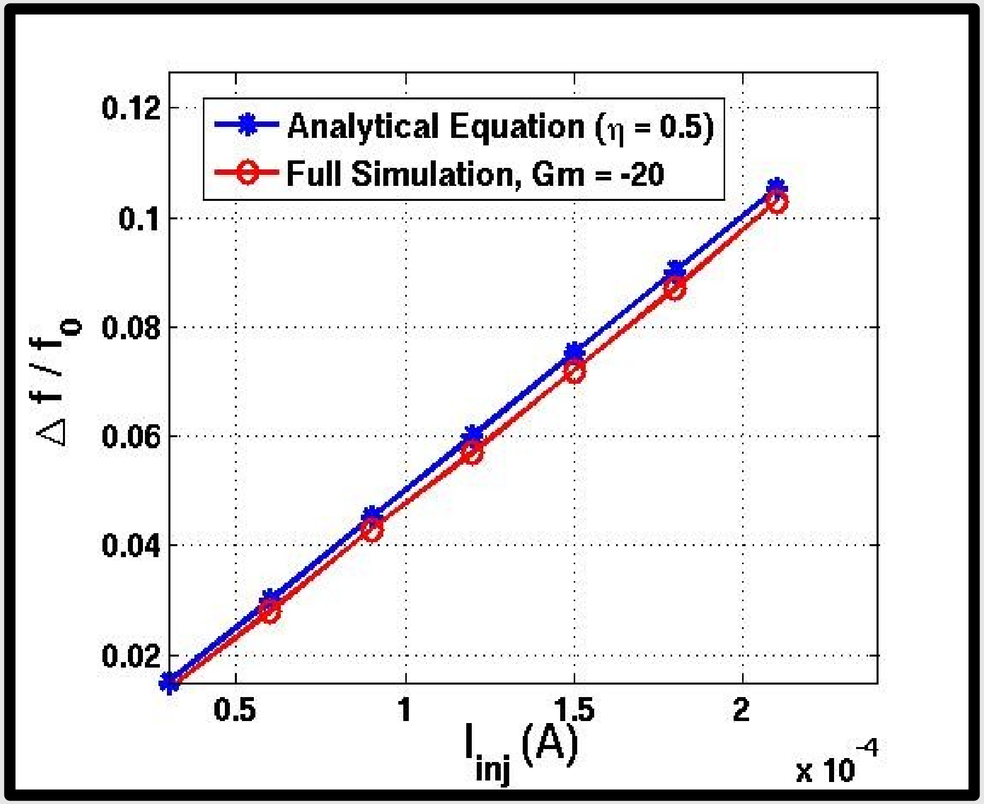
$$f_{inst} = \frac{d\Delta\phi(t)}{dt} + f_1$$

$$f_{inst} = f_0 + f_0 g(\Delta\phi(t))$$

Comparison with Full Simulation



Sinusoidal Injection



Square Wave Injection

Excellent match with the full simulation

Conclusion

- Simple analytical equations for injection locking analysis in ring oscillators
  - maintain Adler like simplicity
  - quick insight into injection locking process via graphical analysis
  - hand analysis of injection locking range for variety of injection signals
  - good match with the full simulation
- Gen-Adler is numerically applicable to any oscillator for injection locking analysis

End