A Moment-Matching Scheme for the Passivity-Preserving Model Order Reduction of Indefinite Descriptor Systems with Possible Polynomial Parts

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**Need for Reduced Order Models (ROMs)**

Reduced-order models (tens of Eqns), easy to simulate...

Accuracy, stability, passivity

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**Step 1. PDE Field Solvers, SPICE Netlist, Measured Data**

- Interconnect
- Vias
- Nanophotonics
- Carbon nanotubes

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**Step 2. Model Order Reduction**

Large LTI descriptor systems (Thousands to Millions Eqns)
Very expensive to simulate...
Need for Passivity Preservation

- The composition of “stable” models can be unstable.
- But the interconnection of “passive models” is stable.

A LTI system is passive if $H(s)$ is positive real:

1. $H(s)$ has no poles in $\Re(s) > 0$
2. $H(s) = H(\overline{s})$ for all $s \in \mathbb{C}$
3. $H(s) + H^*(s) \geq 0$ for all $\Re(s) > 0$

If the models are not passive they can generate energy, and the time-domain simulation may explode in the connected systems-level simulation!
Passive MORs: Moment Matching vs PRBT

Basic MOR Procedures:

Find \( U \) and \( V \) (nxq), and perform projection

\[
\begin{align*}
\frac{d x_r}{d t} &= A_r x_r + B_r u(t), \\
y_r &= C_r x_r + D u
\end{align*}
\]

Small system, \( q \) eqns, \( q \ll n \)

Large system, \( n \) eqns

How to construct \( U \) and \( V \) to preserve passivity?

1. For positive semi-definite systems, moment matching (e.g., PRIMA) \( O(n^2) \)

\[
E + E^T \geq 0, \ A + A^T \leq 0, \ B^T = C
\]

\( U = V \) (congruence transform)

\[
E_r + E_r^T \geq 0, \ A_r + A_r^T \leq 0, \ B_r^T = C_r
\]

2. For indefinite systems, use positive-real balanced truncation, cost \( O(n^3) \)
need solve two generalized algebraic Riccati equations (coming later...)

Moment matching is more efficient thus preferred ...
Limitations of the Conventional Moment Matching

1. Indefinite models: passivity is not guaranteed.
2. If $E$ is singular, the ROM may be very inaccurate.

Consider the following passive descriptor system model

$$E \frac{dx}{dt} = Ax + Bu, \quad y = Cx$$

**Transfer function**

$$H(s) = C (sE - A)^{-1} B = H_{sp}(s) + P(s)$$

- **Strictly proper**
  $$H_{sp}(s) = \sum_{i=1}^{m_1} \frac{F_i}{s - p_i}, \quad H_{sp}(j\infty) = 0$$
  $$P(s) = M_0 + \sum_{k=1}^{m_2} s^k M_k$$

$P(s)=0$: $H(s)$ is strictly proper.

The Krylov-subspace based moment matching generates nonsingular $E_r$, so $H_r(s)$ is strictly proper, and it is inaccurate when $H(s)$ is not strictly proper!!

**Practical Examples:**

RLC circuits with L-I cutsets or C-V loops; discretized PDEs, linearized MOS circuits, MEMs devices
**Polynomial Parts: Simple RLC Examples**

**MNA (modified nodal analysis)**

\[ E = \text{diag} \begin{bmatrix} 0 & C_1 & C_2 & L_1 & 0 \end{bmatrix} \]

\[ A = \begin{bmatrix} -\frac{1}{R_1} - \frac{1}{R_2} & \frac{1}{R_1} & \frac{1}{R_2} & 0 & -1 \\ \frac{1}{R_1} & -\frac{1}{R_1} & 0 & 1 & 0 \\ \frac{1}{R_2} & 0 & -\frac{1}{R_2} & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ P(s) = \frac{1}{R_1} + \frac{1}{R_2} \neq 0 \]

\[ B = C^T = [0 \ 0 \ 0 \ 0 \ 1]^T \]

**Conventional Krylov-subspace moment matching cannot preserve \( P(s) \)!!**
1. Passive Moment Matching for Nonsingular Indefinite Systems
   - How to guarantee accuracy?
     - use moment matching to guarantee accuracy.
   - How to preserve passivity?
     - Relate the MOR method to positive real lemma.

2. Model Reduction for Singular Descriptor Systems
   - How to preserve the possible polynomial part?
   - How to preserve passivity?
Given the passive indefinite model \( E\dot{x} = Ax + Bu, \quad y = Cx + D \)

Assume that \( E \) is nonsingular and \( D + D^T > 0 \)

Proposed MOR Flow (motivated by Bond ICCAD2008)

- **Step 1:** Construct the right projection matrix \( V \), by any existing \( q \)-th order Krylov-subspace moment matching, e.g.

\[
V = \left\{ (s_0 E - A)^{-1} B, \quad (s_0 E - A)^{-1} E (s_0 E - A)^{-1} B, \quad L, \quad \left[ (s_0 E - A)^{-1} E \right]^{q-1} (s_0 E - A)^{-1} B \right\}
\]

- **Step 2:** Compute the positive real observability Gramian \( Q_o \geq 0 \), the stabilizing solution to the GARE:

\[
\tilde{A}^T Q_o E + E^T Q_o \tilde{A} + E^T Q_o \tilde{B} \tilde{B}^T Q_o E + \tilde{C} \tilde{C} = 0, \quad \tilde{B} = B \left( D + D^T \right)^{-\frac{1}{2}} \tilde{C} = \left( D + D^T \right)^{-\frac{1}{2}} C, \quad \tilde{A} = A - \tilde{B} \tilde{C}
\]

- **Step 3:** Compute the left projection matrix \( U \):

\[
U = Q_o EV \left( V^T E^T Q_o EV \right)^{-1}
\]

- **Step 4:** Construct the ROM:

\[
E_r = U^T E V, \quad A_r = U^T A V, \quad B_r = U^T B, \quad C_r = CV
\]
Accuracy

- The first $q$ moments of $H(s)$ are preserved, due to the (block) Krylov-subspace used for constructing $V$

$$V = \left\{ (s_0E - A)^{-1} B, \ (s_0E - A)^{-1} E (s_0E - A)^{-1} B, \ldots, \left[ (s_0E - A)^{-1} E \right]^{q-1} (s_0E - A)^{-1} B \right\}$$

This guarantees a similar accuracy with PRIMA (using $U=V$).

- Furthermore, $U$ provides extra accuracy!

$$U = Q_oEV \left( V^T E^T Q_oEV \right)^{-1} \Rightarrow \text{range}(U) \subset \text{range}(Q_o)$$

Complexity

- Compared with PRBT, our method is 2X faster.

**Our method (one GARE):**

$$\tilde{A}^T Q_o E + E^T Q_o \tilde{A} + E^T Q_o \tilde{B} \tilde{B}^T Q_o E + \tilde{C} \tilde{C} = 0$$

**PRBT (two GAREs):**

$$\begin{cases} \tilde{A}^T Q_o E + E^T Q_o \tilde{A} + E^T Q_o \tilde{B} \tilde{B}^T Q_o E + \tilde{C}^T \tilde{C} = 0 \\ \tilde{A} Q_c E^T + EQ_c \tilde{A}^T + EQ_c \tilde{C}^T \tilde{C} Q_c E^T + \tilde{B} \tilde{B}^T = 0 \end{cases}$$
For the obtained ROM, we can prove that the following LMIs

\[
\left\{ \begin{array}{l}
A^r E_r X E_r + E^r A^r X A^r = -L^r L^r, \\
E^r X B_r - C^r = -L^r W_r, \\
D + D^T \geq W^T W
\end{array} \right.
\]

has a positive semi-definite solution:

\[
X = V^T E^T Q_o E V \geq 0
\]

According to the extended positive real lemma (R. W. Freund and F. Jarre 2004), the constructed ROM is passive.
MOR for Singular Systems

**Singular** Descriptor systems

\[
E \frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du
\]

E is not invertible

**Canonical Form of the Matrix Pencil**

\[
E = T_l \begin{bmatrix} I \\ N \end{bmatrix} T_r, \quad A = T_l \begin{bmatrix} J \\ I \end{bmatrix} T_r
\]

\(T_l\) and \(T_r\) are invertible, \(J\) corresponds to finite system poles, \(N\) is a nilpotent matrix that leads to infinite system poles.

**Spectral Projectors**

\[
P_l = T_l \begin{bmatrix} I \\ 0 \end{bmatrix} T_l^{-1}, \quad P_r = T_r^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} T_r
\]

**Note:** in practice, we do not use the canonical form to compute the spectral projectors; instead, we use the sparse-LU based canonical projector technique to compute the two projector matrices.

**Singular** Descriptor systems

\[ E \frac{dx}{dt} = Ax + Bu, \quad y =Cx + Du \quad \text{E is not invertible} \]

The transfer function of a passive descriptor system:

\[ H(s) = C(sE - A)^{-1}B + D = H_{sp}(s) + M_0 + sM_1 \]

**Passivity Condition for Descriptor Systems:**
The proper part \(H_{sp}(s)+M_0\) is positive real, and \(M_1\) is positive semi-definite.

**Requirements for our moment-matching flow:**

1. Preserve the possible polynomial part \(P(s)\)
2. Reduce \(H_{sp}(s)\), and ensure the passivity of \(H_{sp}(s)+M_0\)

**Our Solution:**

System Decomposition + Passive moment matching for indefinite systems
MOR for Singular Systems

\[
E \frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du \]

E is not invertible

**Step 1: System Decomposition by \( P_r \)**

1. **Define** \( G(s) = C(I - P_r)(sE - A)^{-1}B + D \), then \( H_{sp}(s) + M_0 \) can be realized by a **nonsingular indefinite system** \((E_p, A_p, B_p, C_p, D_p)\):

\[
E_p = EP_r - \alpha A(I - P_r), \quad A_p = A, \quad C_p = CP_r, \quad B_p = B, \quad D_p = G(0), \text{ with } \alpha > 0
\]

2. **Extract** \( M_1 \)

\[
M_1 = \frac{G(s_1) - G(s_2)}{s_1 - s_2}, \text{ with } s_1, s_2 > 0
\]

**Step 2: Reduce the **indefinite** nonsingular model** \((E_p, A_p, B_p, C_p, D_p)\) by our proposed moment matching method, get the ROM

\[
E_{pr} \frac{d\tilde{x}}{dt} = A_{pr} \tilde{x} + B_{pr} u, \quad y = C_{pr} \tilde{x} + D_p u
\]

**Step 3: Reconstruct the **singular** ROM:**

\[
E_r = \begin{bmatrix} E_{pr} & I_m \\ 0 & I_m \\ 0 & 0 \end{bmatrix}, \quad A_r = \begin{bmatrix} A_{pr} & 0 \\ I_m & 0 \\ I_m & M_1 \end{bmatrix}, \quad B_r = \begin{bmatrix} B_{pr} \\ 0 \\ 0 \end{bmatrix}, \quad C_r = \begin{bmatrix} C_{pr}^T \\ -I_m \\ 0 \end{bmatrix}, \quad D_r = D_p
\]

**Note:** The final ROM is passive and the polynomial part \( P(s) \) is preserved.
Numerical Results

Benchmark: an order-1505 RLC MNA model, 5 ports

Verification flow:

- Use sparse LU-based spectral projector
- Use single-point moment matching to construct $V$
- Use Matlab built-in function “gcare” to solve the GARE

System Decomposition

Reduce the proper part $(H_{sp}(s) + M_0)$ by the proposed passive moment matching

Reconstruction to a singular ROM

Compare with PRIMA and PRBT

Compare PRIMA (directly performed on the original MNA model)
Numerical Results

MOR of the Indefinite Proper Part $H_{sp}(s) + M_0$

(a) Frequency response, port(1,1)

(b) MOR error, port(1,1)

Accuracy

Low-freq. band: both moment matching methods are better than PRBT
The proposed method is better than PRIMA in the whole frequency band

CPU Times

<table>
<thead>
<tr>
<th>Model Size</th>
<th>Number of Port</th>
<th>PRIMA (sec)</th>
<th>PRBT (sec)</th>
<th>Proposed (sec)</th>
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<tbody>
<tr>
<td>1505</td>
<td>5</td>
<td>1.76</td>
<td>507.8</td>
<td>243.1</td>
</tr>
</tbody>
</table>
Numerical Results

MOR of the Indefinite Proper Part $H_{sp}(s) + M_0$

The passivity is tested by the generalized Hamiltonian method (GHM).

Z. Zhang and N. Wong, “Passivity test of immitance descriptor systems based on generalized Hamiltonian methods”, IEEE TCAS2, Jan. 2010

Passivity

(a) GHM test result, PRIMA

(b) GHM test result, proposed

PRIMA: Nonpassive ROM

Proposed: Passive ROM
Numerical Results

MOR of whole MNA descriptor-system model

To preserve passivity, PRIMA is directly performed on the original positive semi-definite descriptor system

(a) Frequency response, port(1,1)

The proposed moment-matching method produces a ROM that overlaps the original model in the whole frequency band; but the conventional moment matching (e.g. PRIMA) can’t preserve the polynomial part $P(s)$. 
Thanks for your attention!

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