Balanced Truncation for Time-Delay Systems Via Approximate Gramians

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Outline

• Introduction and Objectives
• Problem Formulation
• Proposed Algorithm
• Experimental Results
• Summary and Conclusions
Introduction:
Model Order Reduction (MOR)

• Exponentially increasing elements are required in the original model for VLSI circuit simulation.

• MOR techniques compact the large model into a reduced-order model.

• Motivation is to reduce the internal complexity while preserving external behaviors.
Introduction:
Model Order Reduction

\[ x(t) = Ax(t) + A_d x(t - d) + Bu(t) \]
\[ y(t) = Cx(t) + Du(t) \quad x \in \mathbb{R}^n \]

Model Reduction
\[ m \ll n \]
\[ y_r \approx y \]

\[ z(t) = A_r z(t) + A_{dr} z(t - d) + B_r u(t) \]
\[ y_r(t) = C_r z(t) + D_r u(t) \quad z \in \mathbb{R}^m \]
Introduction: Time-Delay Systems (TDSs)

• A TDS may arise from a circuit network connected with delay elements such as transmission lines.

• MOR can be performed to compact the model for simulation efficiency.

• It’s the first time to reduce a TDS utilizing the balanced truncation approach.

• It provides a better accuracy than conventional approaches.
Introduction:
Time-Delay Systems (TDSs)
Problem Formulation: Model Reduction of Time-Delay Systems

Original Large TDS: \[
\begin{align*}
x(t) &= Ax(t) + A_d x(t-d) + Bu(t) \\
y(t) &= Cx(t) + Du(t) \quad x \in \mathbb{R}^n
\end{align*}
\]

Reduction by Projection

\[
A_r = V^T AV, \quad A_{dr} = V^T A_d V \\
B_r = V^T B, \quad C_r = CV, \quad D_r = D
\]

\[m \ll n\]

\[y_r \approx y\]

Reduced Small TDS: \[
\begin{align*}
\dot{z}(t) &= A_r z(t) + A_{dr} z(t-d) + B_r u(t) \\
y_r(t) &= C_r z(t) + D_r u(t) \quad z \in \mathbb{R}^m
\end{align*}
\]

- The key is how to construct the projection matrix \(V\) for a time-delay system.
- The difficulty is how to deal with the delay term.
Problem Formulation:
Taylor Expansion based MOR

• Taylor Expansion
  – Taylor expansion of the delay exponential term.
  – Bad approximation.
  – Result in a high-order equivalent LTI system for the Anroldi procedure.

\[
x(t - d) \xrightarrow{s\text{-domain}} X(s)e^{-sd}
\]

\[
e^{-sd} = \sum_{i=0}^{\infty} \frac{(-d)^i}{i!} s^i
\]
Problem Formulation: Padé approximation based MOR

- **Padé approximation**
  - Padé approx. of the delay exponential term.
  - Bad approximation.
  - Result in a high-order equivalent LTI system for the Anroldi procedure / balanced truncation.

\[
x(t - d) \xrightarrow{s-domain} X(s) e^{-sd}
\]
\[
e^{-sd} \approx \frac{P_d(s)}{Q_d(s)} = \tilde{C}(sI - \tilde{A})^{-1} \tilde{B} + \tilde{D}
\]
Problem Formulation: MOR via Balanced Truncation

- Balanced Truncation
  - Gramians of Time-Delay Systems.
    - S. Yi, ‘Controllability and observability of systems of linear delay differential equations via the matrix Lambert W function’, 2008
    - The state solution involves Lambert W functions.

\[ \dot{x}(t) = Ax(t) + A_d x(t - d) + Bu(t), \quad y(t) = Cx(t) \]

\[ x(t) = \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{S_k (t-\tau)} C_k^N Bu(\tau) d\tau, \quad \text{where} \quad S_k = \frac{1}{d} W_k (A_d d Q_k) + A \]

\[ P = \int_0^\infty \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^N B B^T \left( \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^N \right)^T dt \]

\[ Q = \int_0^\infty \left( \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^N \right)^T C^T C \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^N dt \]

The Gramians of TDSs
Problem Formulation:
Lyapunov-Type Equations for TDSs

• By defining

\[ w_{ij} = e^{S_i t} C_i^N B B^T (e^{S_j t} C_j^N)^T \]

\[ P_{ij} = \int_0^\infty w_{ij} dt \]

• The controllability Gramian can be rewritten as

\[ P = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \int_0^\infty w_{ij} dt = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} P_{ij} \]

• Lyapunov-type equations for the controllability Gramian components

\[ S_i P_{ij} + P_{ij} S_j^T + C_i^N B B^T (C_j^N)^T = 0 \]
Problem Formulation:
Gramians in the s-domain

• The Lyapunov-type equations are hard to be solved
  – It involves Lambert W functions.
  – Convergence of the branches.
  – High computational cost.

• Gramians in the s-domain [S. Yi, 2008]
  – The formulations are much simpler.
  – Use the Poor Man’s TBR to approximate.

\[
P = \int_{-\infty}^{\infty} (sI - A - A_d e^{-sd})^{-1} B B^T (sI - A - A_d e^{-sd})^{-H} ds
\]
\[
Q = \int_{-\infty}^{\infty} (sI - A - A_d e^{-sd})^{-H} C^T C (sI - A - A_d e^{-sd})^{-1} ds
\]
Problem Formulation: 
the Poor Man’s TBR

• The Poor Man’s TBR for LTI Systems [J. R. Phillips, 2005]
  – It utilizes Laplace transform of the Gramians.

\[
P = \int_{-\infty}^{\infty} (sI - A)^{-1} B B^T (sI - A)^{-H} \, ds
\]

\[
Q = \int_{-\infty}^{\infty} (sI - A^T)^{-1} C^T C (sI - A^T)^{-H} \, ds
\]

– It uses finite summation to approximate the infinite integration.

\[
z_{ck} = (j \omega_k I - A)^{-1} B \quad z_{ok} = (j \omega_k I - A^T)^{-1} C^T
\]

\[
\tilde{P} = \sum_k z_{ck} z_{ck}^H \quad \tilde{Q} = \sum_k z_{ok} z_{ok}^H
\]
Proposed Algorithm: TBR using the Approximate Gramians

- Balanced Truncation for Time-Delay Systems
  - Calculate the Gramians in the s-domain.
    \[
    P = \int_{-\infty}^{\infty} (sI - A - A_d e^{-sd})^{-1} B B^T (sI - A - A_d e^{-sd})^{-H} ds
    \]
    \[
    Q = \int_{-\infty}^{\infty} (sI - A - A_d e^{-sd})^{-H} C^T C(sI - A - A_d e^{-sd})^{-1} ds
    \]
  - Use finite summation to approximate the infinite integration.
    \[
    z_{ck} = (j\omega_k I - A - A_d e^{-j\omega_k d})^{-1} B, \quad z_{ok} = (j\omega_k I - A - A_d e^{-j\omega_k d})^{-H} C^T
    \]
    \[
    \tilde{P} = \sum_k z_{ck} z_{ck}^H, \quad \tilde{Q} = \sum_k z_{ok} z_{ok}^H
    \]
Proposed Algorithm: Practical Implementations for TDSs

- **Time-Delay Systems**
  - Multiple delays
  - Descriptor systems

\[ E\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_{d_i} x(t - d_i) + Bu(t) \]

\[ z_{ck} = (j\omega_k E - A - \sum_{i=1}^{m} A_{d_i} e^{-j\omega_k d_i})^{-1} B \]

\[ z_{ok} = (j\omega_k E - A - \sum_{i=1}^{m} A_{d_i} e^{-j\omega_k d_i})^{-H} C^T \]

\[ \tilde{P} = \sum_k z_{ck} z_{ck}^H \quad \tilde{Q} = \sum_k z_{ok} z_{ok}^H \]
Proposed Algorithm: Balanced Truncation for TDSs

- The Proposed MOR Procedures for TDSs
  1. Select the frequency-sampling points uniformly within the frequency band of interest.
  2. Compute the approximate Gramians in the s-domain.
  3. Use canonical TBR to reduce the model.
Experimental Results: A Time-Delay System Example

- A Time-Delay System Example
  - A 3-port linear interconnect network.
  - 70 lossless 3-conductor transmission lines.
  - The resulting TDS is of order 1098.
  - The delays are in the magnitude of nanoseconds.
  - The delay effects in the frequency response emerge from around 1GHz.
Experimental Results: A Time-Delay System Example

- The Reduced-Order Models
  - By proposed TBR: it’s of order 220.
    - Gramians are approximated by using 50 sampling points distributed uniformly in logarithm scale within the frequency range of interest.
  - By moment matching: it’s of order 231.
    - 2nd-order Taylor expansion.
    - Matching the first 77 moments.
  - By Padé approx. based method: it’s out of memory
    - Only using 2nd-order Padé approximation.
Experimental Results:
Frequency Response

![Graph showing frequency response](image)

- **Magnitude (dB) of H(1)**
- **Frequency (Hz)**
- **Relative Error (dB)**

Legend:
- **Original**
- **Proposed TBR**
- **Moment Matching**
Experimental Results:
Time-Domain Response
Experimental Results: Comparison of Computational Times

- **Speedup of Reduced-Order Models**
  - ROMs by different MORs are of similar order.
  - Around 10X speedup in transient simulation.

- **Building Time of Various MORs**
  - By proposed TBR: 71.07 seconds.
  - By moment matching: 43.51 seconds.
  - By Padé-approx. based method: out of memory.

- **Advantage of the Proposed Algorithm**
  - Higher accuracy with comparable efficiency
Summary and Conclusions

• We proposed and implemented a TBR-type MOR for time-delay systems.

• It’s the first time to perform MOR on TDSs via a Gramian-based approach.

• Higher accuracy with comparable efficiency than conventional approaches.

Thanks for your time and attention!