Efficient Synthesis of Quantum Circuits Implementing Clifford Group Operations

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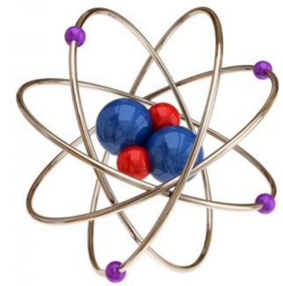
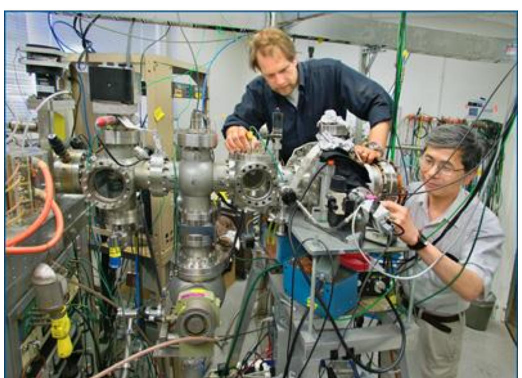
Outline

- Introduction
  - Background and Motivation
  - Synthesis of Quantum Circuits
  - Clifford Group
- Synthesis of Clifford Group Circuits
- Experimental Results and Conclusion

Motivation



- Conventional techniques (e.g. CMOS) approach atomic border
- Promising alternative: quantum computation
  - exploiting quantum-mechanical phenomena
  - fast algorithms for important problems

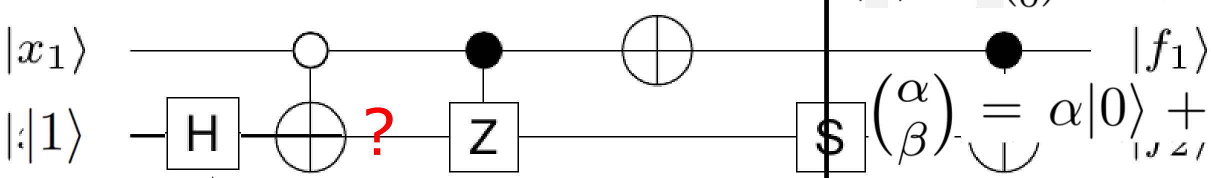


Quantum Computation

- Quantum Bits (*qubits*)
  - basis states $|0\rangle$ and $|1\rangle$
  - superposition $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ $\quad |\alpha|^2 + |\beta|^2 = 1$
- Quantum Circuits

$$|0\rangle \equiv \binom{1}{0} \qquad |1\rangle \equiv \binom{0}{1}$$

$$\binom{\alpha}{\beta} = \alpha|0\rangle + \beta|1\rangle$$



transformation matrix

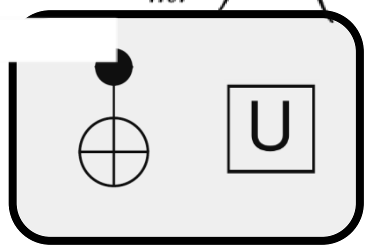
$$H|1\rangle = \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Synthesis of Quantum Circuits

- Synthesis
  - Decomposition of transformation matrix
  - using only gates from a restricted library

$$\overbrace{\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & \\ \vdots & \vdots & \ddots & \\ x_{m1} & & & x_{mn} \end{pmatrix}}^{X} = \overbrace{\begin{pmatrix} u_{11} & \cdots & u_{1r} \\ \vdots & \ddots & \\ u_{m1} & & u_{mr} \end{pmatrix}}^{U} \overbrace{\begin{pmatrix} s_{11} & 0 & \cdots \\ 0 & \ddots & \\ \vdots & & s_{rr} \end{pmatrix}}^{S} \overbrace{\begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \\ v_{r1} & & v_{rn} \end{pmatrix}}^{V}$$

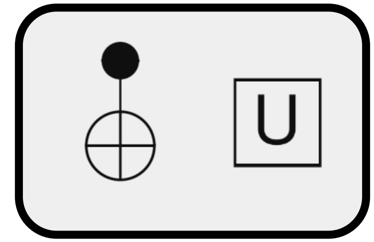
- Quantum gate libraries:
  - CNOT and arbitrary 1-qubit gates



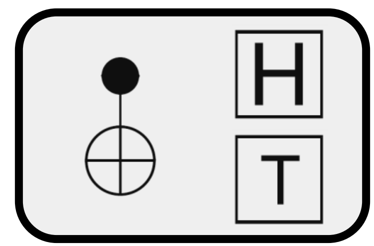
  - in practice, gates have to be approximated
  - fault tolerance?

Quantum Gate Libraries (continued)

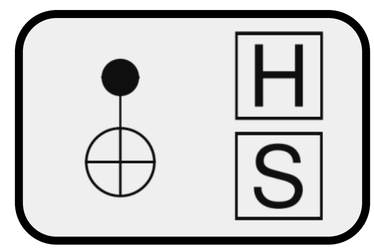
- CNOT and arbitrary 1-qubit gates
  - universal, but hard to realize



- CNOT, Hadamard and T gates
  - universal and fault-tolerant
  - approximation overhead

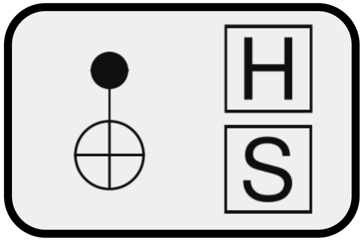


- CNOT, Hadamard and $S=T^2$ gates
  - ➢ *Clifford group*



Clifford Group Operations

- Gate library



- Cover core aspects of quantum functionality
  - (superposition, entanglement, phase shifts)
- Not universal, but sufficient for many applications
  - error-correcting codes (stabilizer circuits)
  - quantum teleportation
  - dense quantum coding

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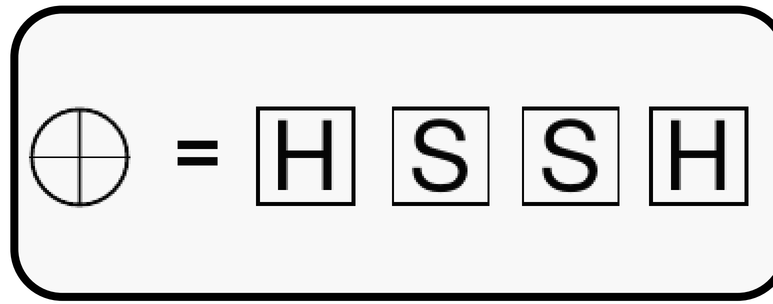
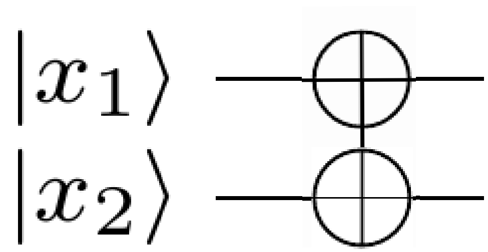
- Synthesis of Clifford Group Circuits
  - Effects on Transformation matrices
  - General idea
  - Running example
- Experimental Results and Conclusion

Effects on Transformation Matrices

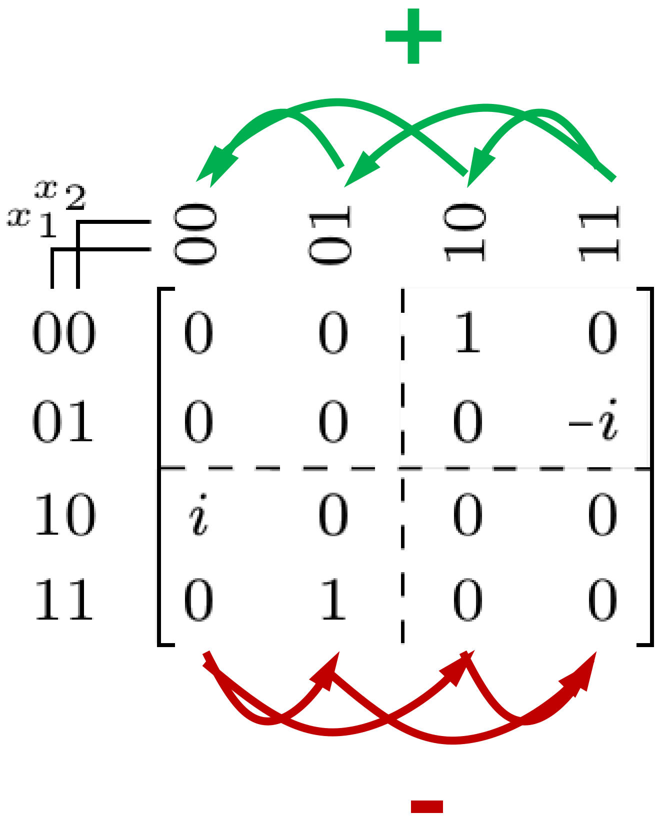
$$
\begin{array}{c|cc:cc}
x_1 x_2 & 00 & 01 & 10 & 11 \\
\hline
00 & 0 & 0 & 1 & 0 \\
01 & 0 & 0 & 0 & -i \\
\hdashline
10 & i & 0 & 0 & 0 \\
11 & 0 & 1 & 0 & 0
\end{array}
$$

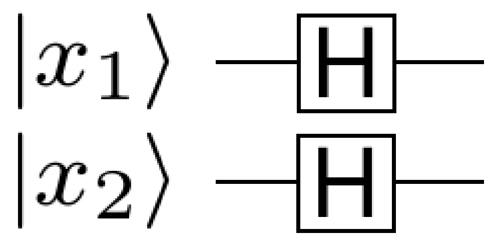




- ➢ Permute columns

Effects on Transformation Matrices



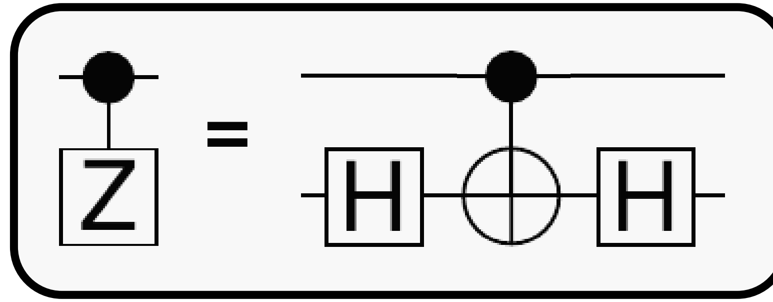


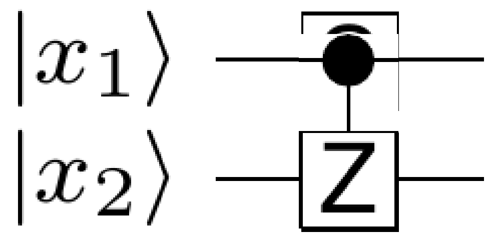
- Combine columns

Effects on Transformation Matrices

Multiply by −1

$$
\begin{array}{c|cccc}
{}_{x_1}^{x_2} & 00 & 01 & 10 & 11 \\
\hline
00 & 0 & 0 & 1 & 0 \\
01 & 0 & 0 & 0 & -i \\
10 & i & 0 & 0 & 0 \\
11 & 0 & 1 & 0 & 0
\end{array}
$$

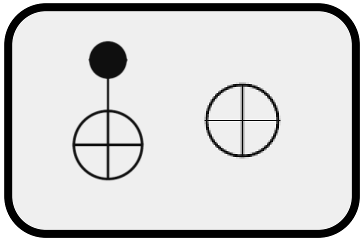




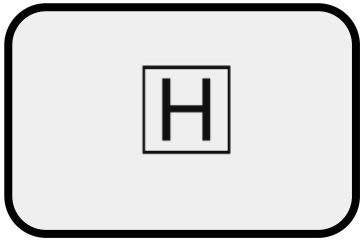
- Modify phase

Effects on Transformation Matrices

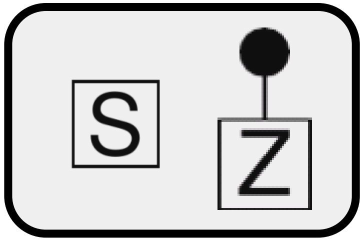
- Summary
  - permute columns



  - combine columns

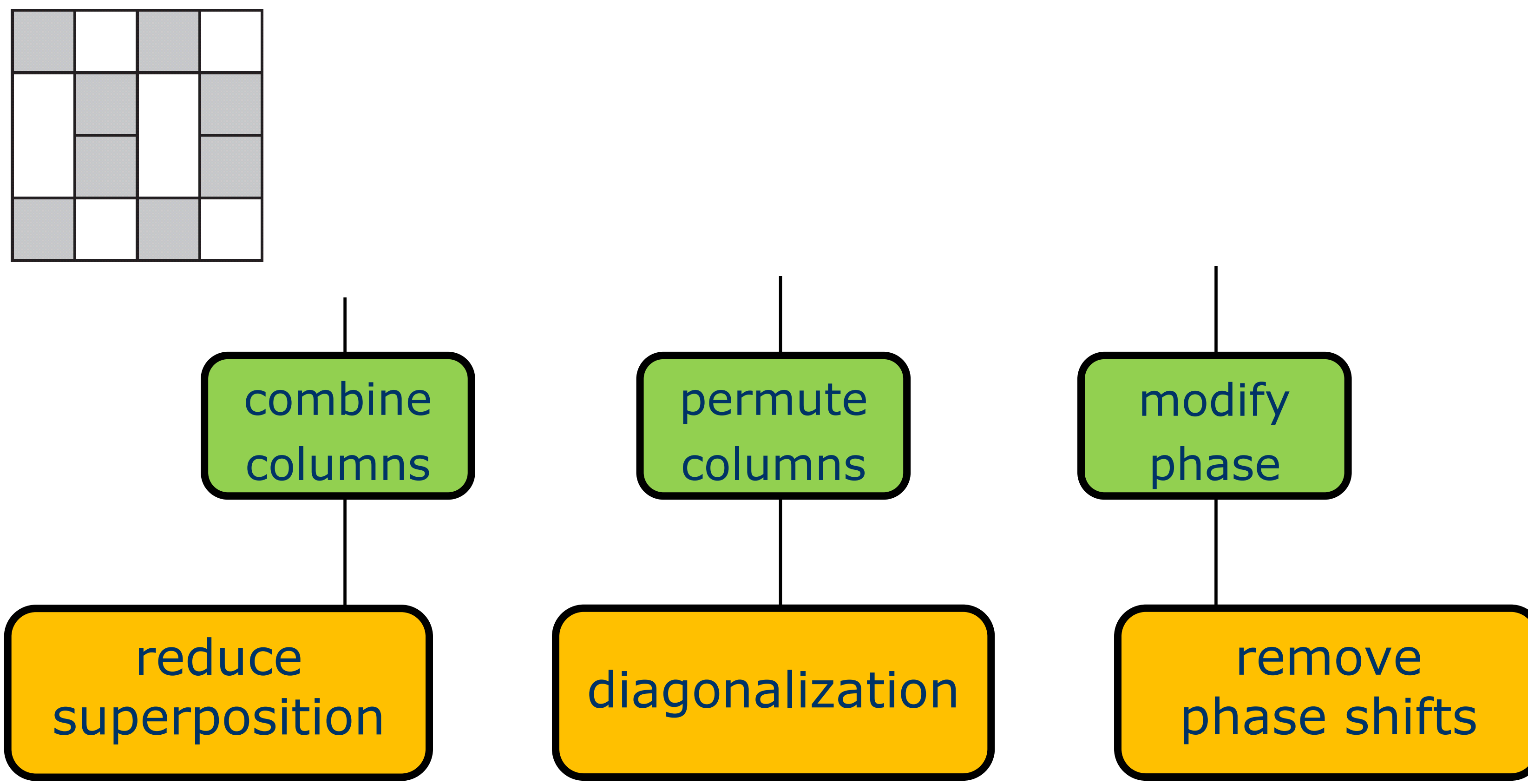


  - modify phase



Synthesis of Clifford Group Circuits

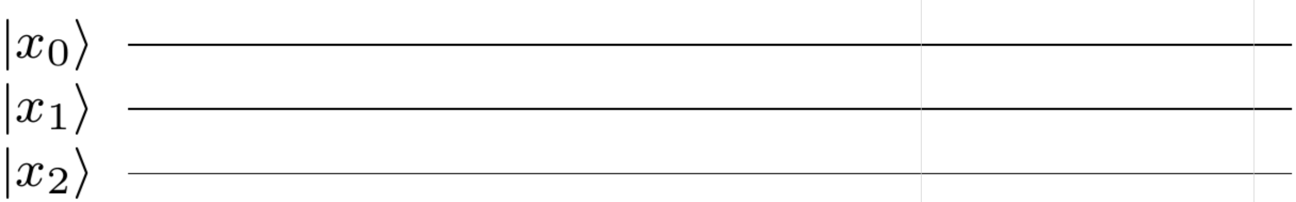
- General Idea



Running example (Step 1)

$x_0 x_1 x_2$	000	001	010	011	100	101	110	111
000	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{i}{2}$	0	$\frac{-i}{2}$
001	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{-i}{2}$	0	$\frac{i}{2}$	0
010	$\frac{1}{2}$	0	$\frac{-1}{2}$	0	$\frac{-i}{2}$	0	$\frac{-i}{2}$	0
011	0	$\frac{-1}{2}$	0	$\frac{1}{2}$	0	$\frac{-i}{2}$	0	$\frac{-i}{2}$
100	0	$\frac{-i}{2}$	0	$\frac{-i}{2}$	0	$\frac{-1}{2}$	0	$\frac{1}{2}$
101	$\frac{-i}{2}$	0	$\frac{-i}{2}$	0	$\frac{1}{2}$	0	$\frac{-1}{2}$	0
110	$\frac{-i}{2}$	0	$\frac{i}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
111	0	$\frac{i}{2}$	0	$\frac{-i}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$

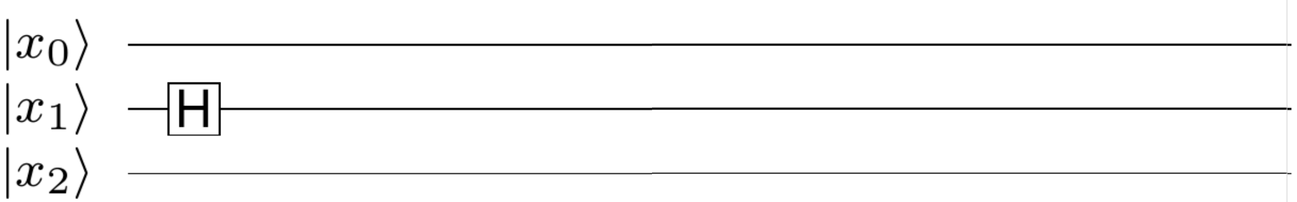
- Hadamard gate can be applied directly at qubit x_1



Running example (Step 1)

$x_0x_1x_2$	000	001	010	011	100	101	110	111
000	0	$\frac{1}{\sqrt{2}}$	0	0	0	$\frac{-1}{\sqrt{2}}$	0	0
001	$\frac{1}{\sqrt{2}}$	0	0	0	$\frac{1}{\sqrt{2}}$	0	0	0
010	0	0	$\frac{1}{\sqrt{2}}$	0	0	0	$\frac{1}{\sqrt{2}}$	0
011	0	0	0	$\frac{-1}{\sqrt{2}}$	0	0	0	$\frac{1}{\sqrt{2}}$
100	0	$\frac{-i}{\sqrt{2}}$	0	0	0	$\frac{-i}{\sqrt{2}}$	0	0
101	$\frac{-i}{\sqrt{2}}$	0	0	0	$\frac{i}{\sqrt{2}}$	0	0	0
110	0	0	$\frac{-i}{\sqrt{2}}$	0	0	0	$\frac{i}{\sqrt{2}}$	0
111	0	0	0	$\frac{i}{\sqrt{2}}$	0	0	0	$\frac{i}{\sqrt{2}}$

- Hadamard gate can **not** be applied directly
- Rearrange columns first
- Align phase shifts
- ➢ H gate can be applied at qubit x_0

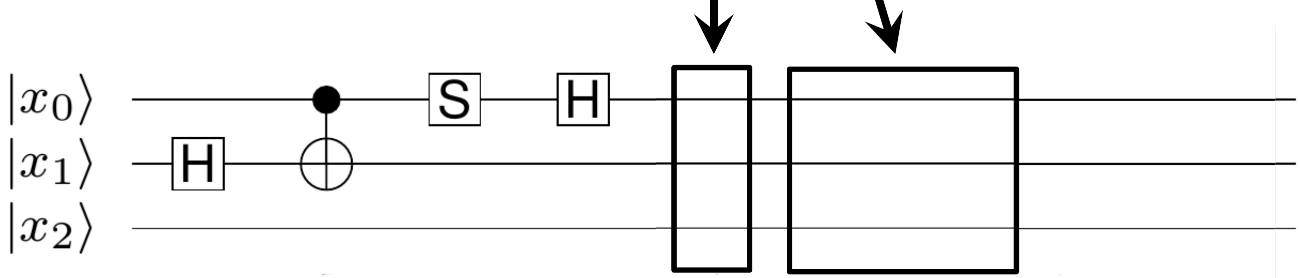


Running example (Step 2)

$x_0 x_1 x_2$	000	001	010	011	100	101	110	111
000	0	0	0	0	0	1	0	0
001	1	0	0	0	0	0	0	0
010	0	0	1	0	0	0	0	0
011	0	0	0	0	0	0	0	-1
100	0	$-i$	0	0	0	0	0	0
101	0	0	0	0	$-i$	0	0	0
110	0	0	0	0	0	0	$-i$	0
111	0	0	0	i	0	0	0	0

- No more superposition
- Diagonalize

$$f_{x_0} = x_0 \oplus x_2$$
$$f_{x_1} = x_1$$
$$f_{x_2} = x_1 \oplus x_2$$

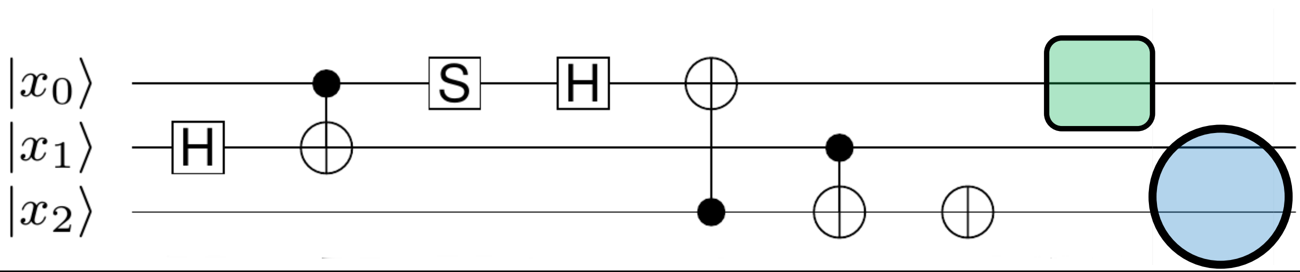


Running example (Step 3)

$x_0 x_1 x_2$	000	001	010	011	100	101	110	111
000	1	0	0	0	0	0	0	0
001	0	1	0	0	0	0	0	0
010	0	0	1	0	0	0	0	0
011	0	0	0	1	0	0	0	0
100	0	0	0	0	1	0	0	0
101	0	0	0	0	0	1	0	0
110	0	0	0	0	0	0	1	0
111	0	0	0	0	0	0	0	1

- Remove phase shifts

✓ Done!



Complexity and Convergence

- Complexity
  - Matrices grow exponentially!
  - Use of adequate data structures
    (e.g. QMDD, QuIDD)

- Convergence
  - Guaranteed by special properties of Clifford Groups
  - Terminates after $\mathcal{O}(n^2)$ steps

Experimental Results

Benchmark	#Qubits	#CNOTS [A1]	#CNOTS [A2]	#CNOTS proposed	#one-qubit gates [A2]	#one-qubit gates proposed

[A1] Shende, Bullock, and Markov, "Synthesis of quantum-logic circuits," IEEE Trans. on CAD, vol. 25, no. 6, pp. 1000–1010, 2006.

[A2] Saeedi, Arabzadeh, Zamani, and Sedighi, "Block-based quantum-logic synthesis," Quantum Information & Computation, vol. 11, no. 3&4, 2011.

Conclusion

- Clifford group
  - ✓ Restricted, but powerful
  - ✓ Fault-tolerant gate library
- Synthesis results
  - ✓ Exploits effects of group generators
  - ✓ Significantly better results than generic approaches

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