Closed form solution for optimal buffer sizing

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Outline

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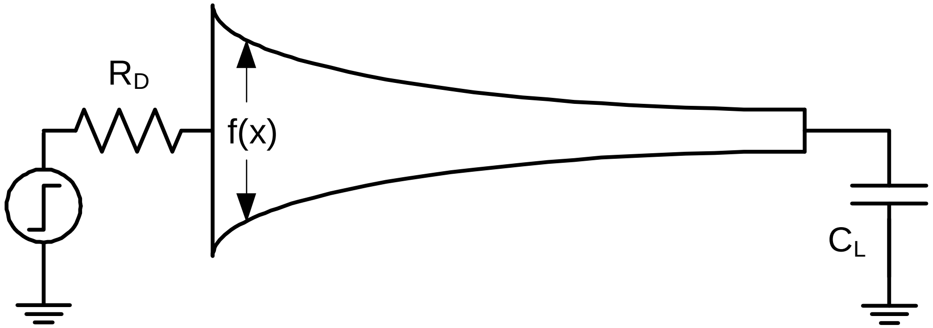
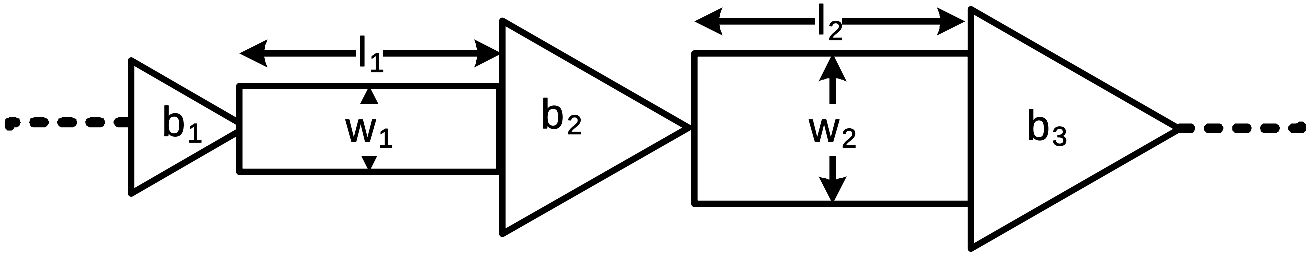
- Assumptions
- Recurrence relation
- Closed form solution
- Testing procedure and results
- Conclusion

Some delay minimization techniques

- Wire sizing

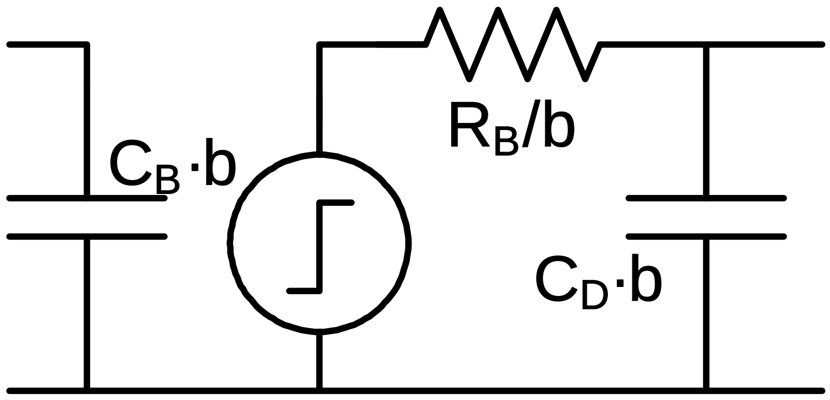


- Buffer insertion



Buffer sizing

- Can also use buffers of different sizes
- Larger buffer has smaller output resistance, but represents larger capacitive load to previous wire segment
- Use a very simple buffer model (buffer of size b)

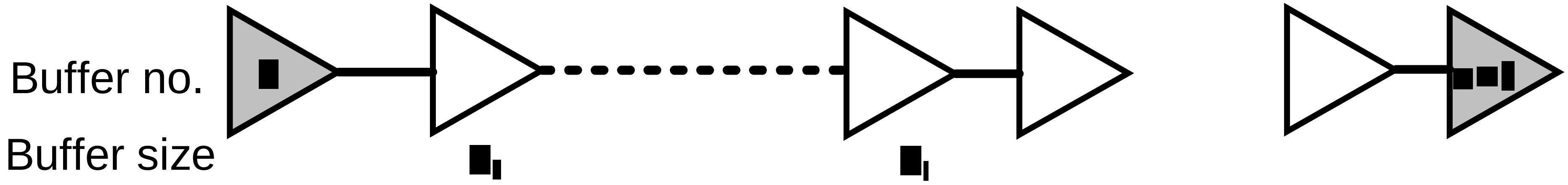


Model assumptions

- Single interconnection wire with fixed width
- No wire sizing allowed
- Elmore delay model
- Use π model for wire segments
- Allow buffer sizing
- Number of buffers given
- Buffers are equally spaced along the wire
- Driver (resistance) and load (capacitance) are buffers of fixed size

Interconnect with buffers

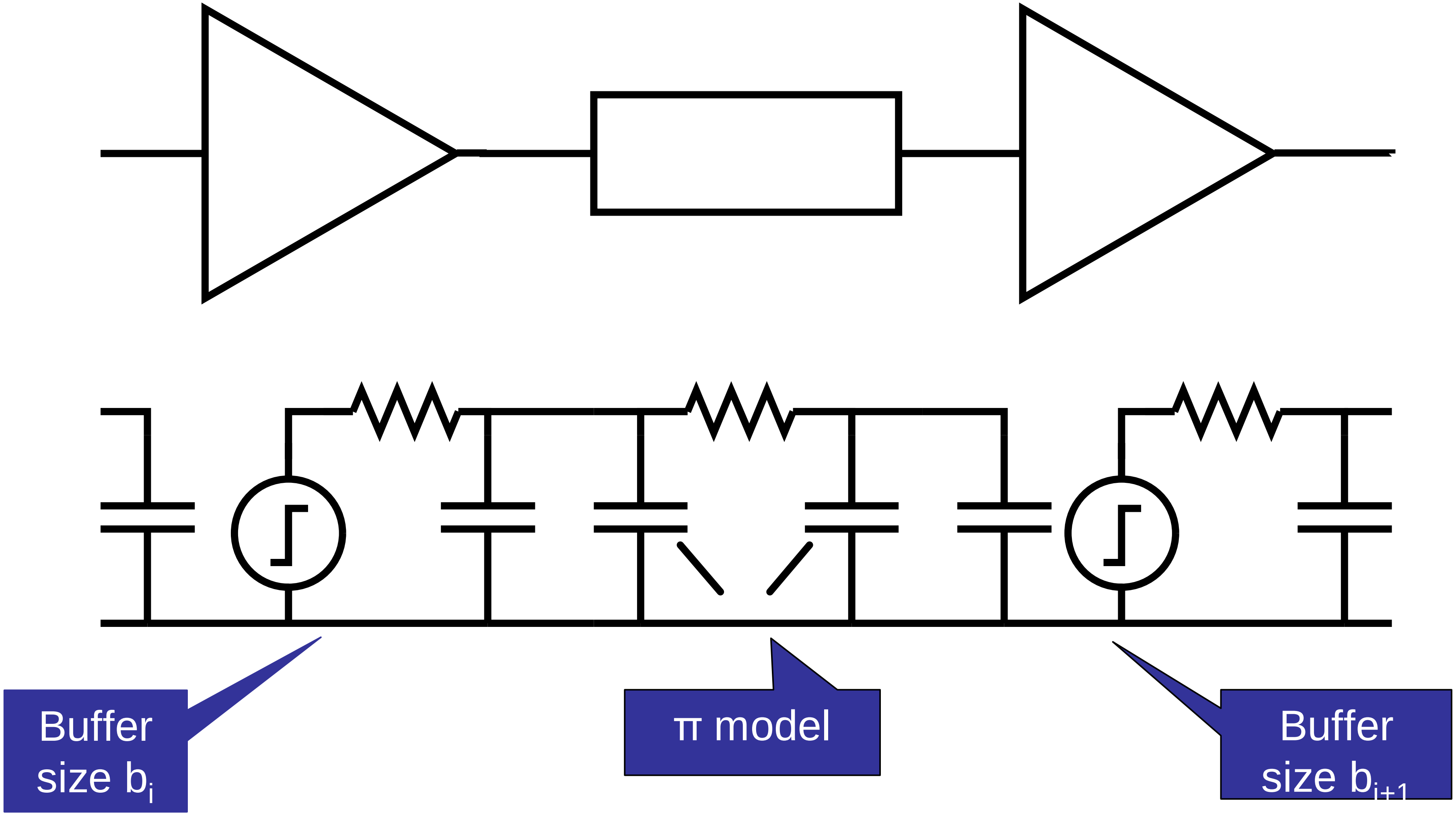
- Source and sink buffers are given



Central question:

What are the optimal buffer sizes b_i to minimize the overall delay? Is there a closed form solution?

Equivalent circuit with buffers i and i+1



Calculate a recurrence relation

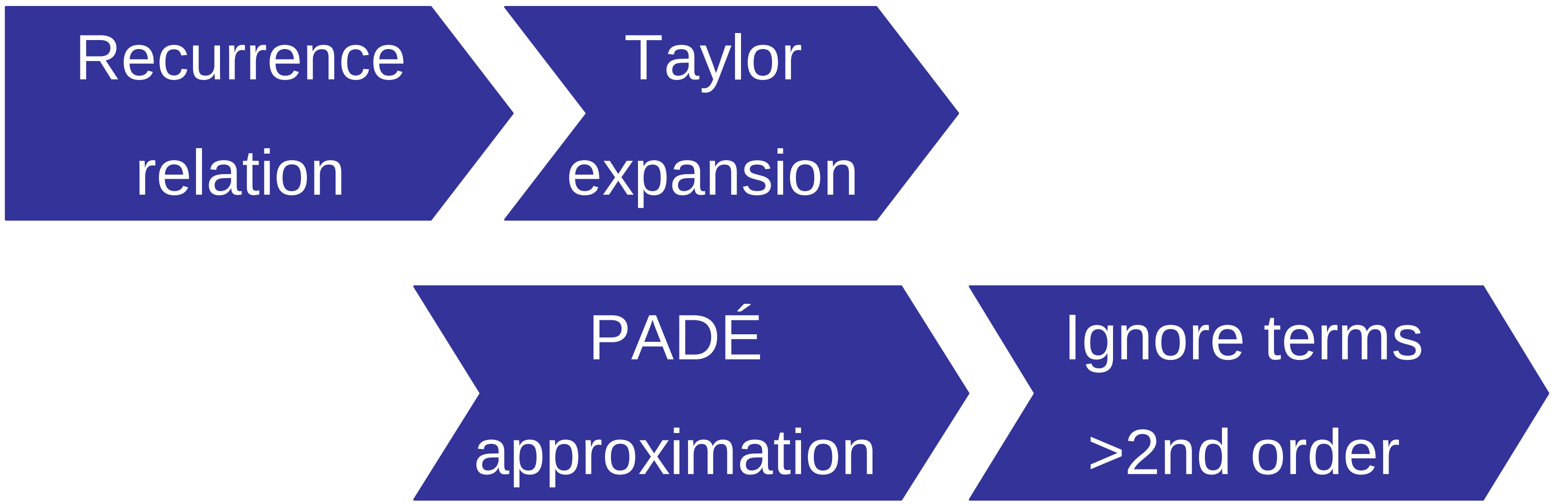
- Calculate Elmore delay ED_i between buffers i and i+1
- Elmore delay from source to sink is $$ED = \sum_{i=0}^{n} ED_i$$
- Find optimal buffer size b_i by setting $$\frac{\partial ED}{\partial b_i} = 0$$

$$b_i = \sqrt{\frac{b_{i+1} + \frac{C}{C_B(1+n)}}{b_{i-1}^{-1} + \frac{R}{R_B(1+n)}}} = \sqrt{\frac{b_{i+1} + \frac{\alpha}{(1+n)}}{b_{i-1}^{-1} + \frac{\beta}{(1+n)}}}$$

$$b_0 = \lambda,\ b_{n+1} = \mu$$

Closed form solution I

- Want a function f(x) that gives optimal buffer size when being evaluated at position x.
- Source is at x=0 and sink is at x=1
- Has to match boundary conditions f(0)=ë and f(1)=µ



Closed form solution II

- Intermediate result
  - First order ordinary differential equation with two boundary conditions
  - c is integration constant, easy calculation

$$f'^2 = \left(\frac{df}{dx}\right)^2 = 2 \cdot n \cdot \beta \cdot f^3 + c \cdot f^2 + 2 \cdot n \cdot \alpha \cdot f$$

- ODE can be solved with Weierstrass elliptic function $\wp$
  - Important property:

$$\wp'(z)^2 = 4\wp^3 - g_2\wp(z) - g_3$$

Closed form solution III

- Transformation of variables gives

$$f'(x)^2 = 4f^3 - g_2 f(x) - g_3$$

- $g_2 = g_2(\alpha, \beta, c, n)$ $g_3 = g_3(\alpha, \beta, c, n)$ are real constants
- ODE can be solved by setting

$$f(x) = \wp(x, g_2, g_3)$$

Closed form solution IV

- According to Weierstrass, the solution f(x) can be given in terms of $\wp$, $\wp'$.

- We denote $h(x) = 2\beta \cdot n \cdot x^3 + c \cdot x^2 + 2\alpha \cdot n \cdot x$
- The closed form solution is given by

$$f(x) = \lambda + \frac{\sqrt{h(\lambda)}\wp'(x) + \frac{1}{2}h'(x)\left(\wp(x) - \frac{1}{24}h''(\lambda)\right) + \frac{1}{24}h(\lambda)h'''(\lambda)}{2\left(\wp(x) - \frac{1}{24}h''(\lambda)\right)^2}$$

Closed form solution V

- Input:
  - Physical parameters of the wire (resistance, capacitance, geometrical dimensions)
  - Physical parameters of a unit size buffer (input capacitance, output resistance, intrinsic delay)
  - No. of buffers n

- Evaluate f(x) at equally spaced points to get optimal buffer size:

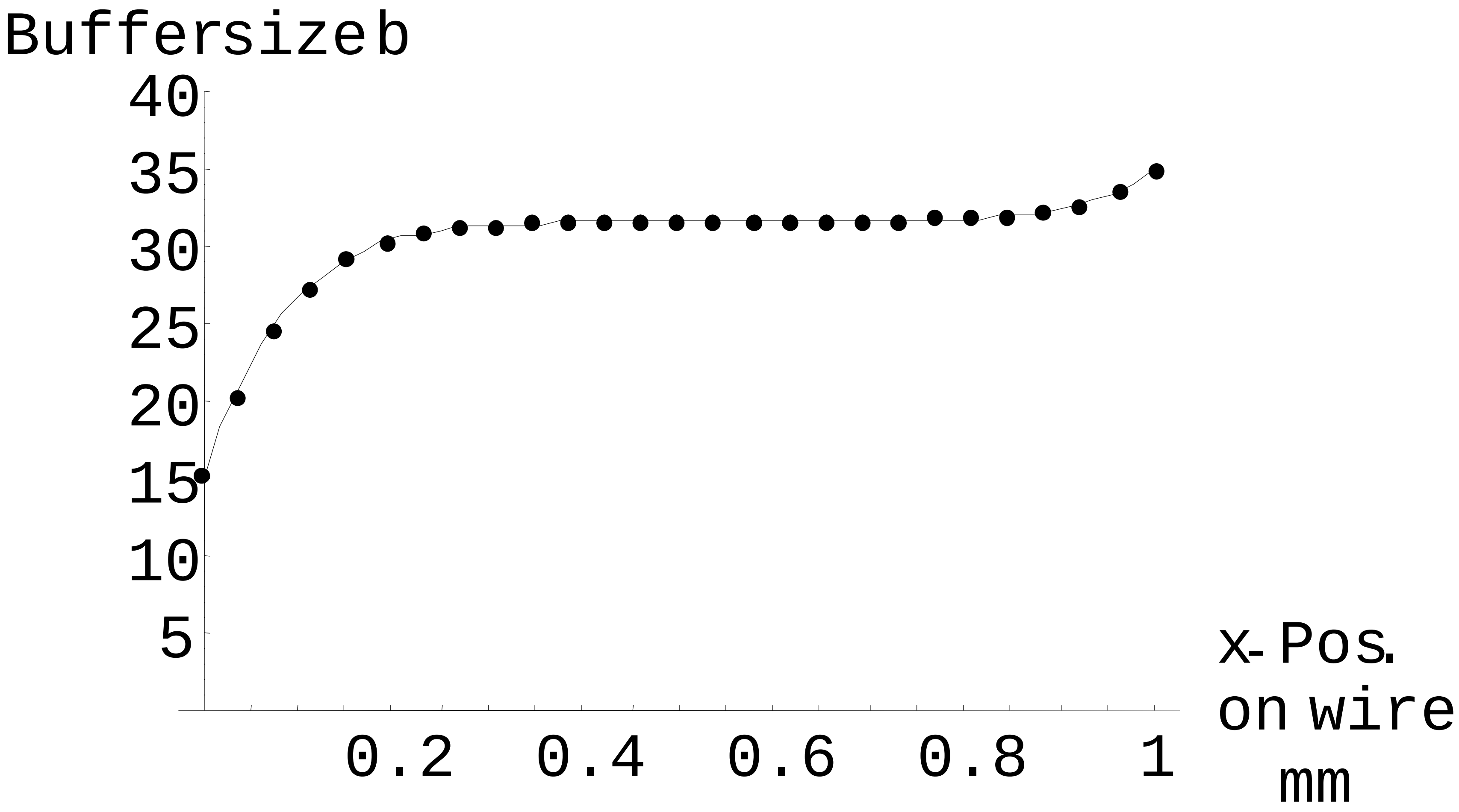
$$f(x) = f(x_i),\ x_i = \frac{i}{n+1}, 1 \le i \le n$$

Testing procedure

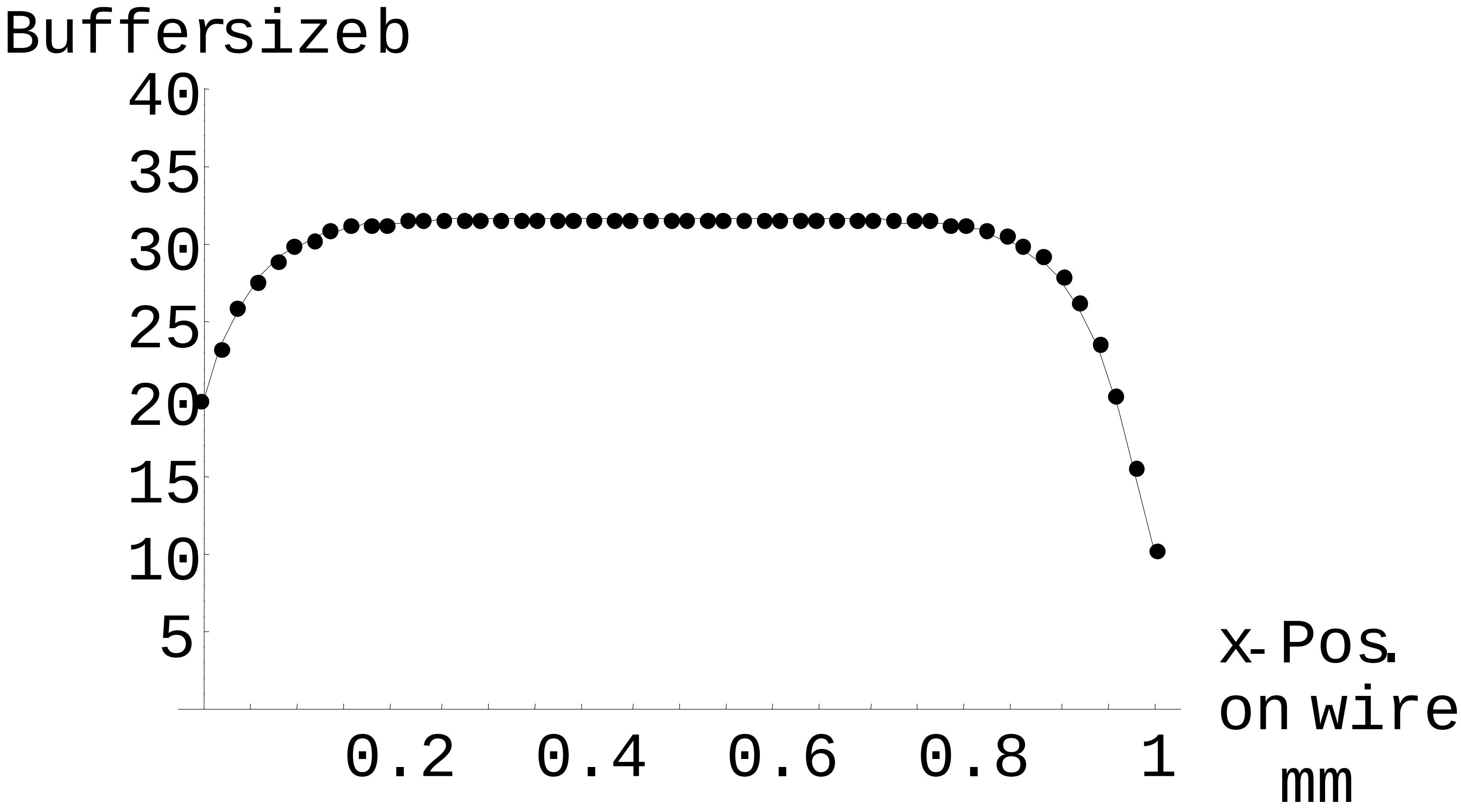
- A) Iteratively
  - Fix every buffer size except buffer i
  - Apply recurrence relation, giving buffer size b_i as a function of b_{i-1} and b_{i+1}
  - Proceed to next buffer
  - Repeat until buffer sizes converge
- B) Use closed form solution
  - Weierstrass function available in Mathematica or Mathlab

- Calculate the Elmore delay for both sets of buffer sizes
  - Compare the relative difference

Example 1: 25 buffers



Example 2: 50 buffers



Results

- Achievements:
  - New closed form solution
  - Buffer sizes derived from closed form solution match those from the iterative method very well
  - Discrepancy diminishes as n increases

- Limitations:
  - Only line topology allowed
  - Equally spaced buffers - does not consider obstacles

Conclusion

- Discussed the underlying model
- Derived recurrence relation
- Obtained a closed form solution in terms of Weierstrass elliptic function
- Discussed some results and consequences

Thank you for your attention!

Substitute to apply Weierstrass function

- Start with $$2\beta n f^3 + c f^2 + 2\alpha n t$$
- Perform substitution $$\hat{f} = \frac{2}{n\beta} f - \frac{c}{6n\beta}$$
- Result: $$4\hat{f}^3 - \underbrace{\left(\frac{c^2}{12} - \alpha\beta n^2\right)}_{g_2} \hat{f} - \underbrace{\left(\frac{1}{12}\alpha\beta n^2 c - \frac{c^3}{216}\right)}_{g_3}$$

Elliptic Integrals

- Some apparently simple integrals have no simple solution.

$$I(x) = \int_c^x R\left(t, \sqrt{P(t)}\right) dt$$

  - R(.) is a rational function of two arguments
  - P is a polynomial of third or fourth degree
  - c is a constant
- Such integrals are called elliptic integrals, can be solved with elliptic functions
  - Jacobi´s elliptic and theta functions
  - General theory: Weierstrass elliptic function

Elliptic Functions

- ω_1 and ω_2 are complex numbers (half-periods) generating a lattice Γ in the complex plain
- A function f is called doubly-periodic if it fulfills

$$f(z) = f(z+2\omega_1) + f(z+2\omega_2), \omega_1 \neq \omega_2, \forall z$$

- Elliptic functions are doubly-periodic functions with certain properties
- For each lattice Γ there is always a non-constant elliptic function $\wp$, the Weierstrass elliptic function, defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{\gamma \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right)$$

Weierstrass elliptic function I

- Consequently, $\wp$ has three arguments:

$$\wp(z) = \wp(z; \omega_1, \omega_2)$$

- Most important property for us:

$$\wp'(z)^2 = 4\wp^3 - g_2\wp(z) - g_3$$

  - g_2 and g_3 are constants that depend on the half-periods ω_1 and ω_2. They are called invariants.
  - Can also write $\wp(z) = \wp(z; g_2, g_3)$

- It is always possible to construct appropriate half-periods ω_1 and ω_2, given the invariants g_2 and g_3.

Weierstrass elliptic function II

- For example, we can easily give the solution to this elliptic integral (here g_2 and g_3 are taken as constants)

$$u = \int_y^\infty \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}} \longrightarrow y = \wp(u; g_2, g_3)$$

- Proof: Integrate the Weierstrass differential equation

- There are (complicated) closed form solutions involving $\wp$ and $\wp'$ for other endpoints of the integral.

Continous version of the recurrence relation

- Replace b_i by $f(x_i)$

$$f(x_i)^2 = \frac{f(x_{i+1}) + \frac{\alpha}{n+1}}{f(x_{i-1})^{-1} + \frac{\beta}{n+1}}$$

- Substitute

$$x_i = x, \Delta x = \frac{1}{n+1}$$

$$x_{i+1} = x + \Delta x, x_{i-1} = x - \Delta x$$

$$\frac{1}{n} \cdot \frac{1}{n+1} \approx \Delta x^2$$

- Result

$$f(x)^2 = \frac{f(x+\Delta x) + \Delta x^2 n\alpha}{f(x-\Delta x)^{-1} + \Delta x^2 n\beta}$$

Taylor expansion + Pade approximation

- Substitute $f(x) = e^{g(x)}$
- Perform Taylor expansion, result: rational function

$$f(x)^2 = e^{2g(x)} = \frac{P(\Delta x)}{Q(\Delta x)} = A(\Delta x)$$

- Compare coefficients to get

$$A(\Delta x) = a_0 + a_1 \Delta x + a_2 \Delta x^2 + O(\Delta x^3)$$

$$= e^{2g(x)} (1 + \Delta x^2 \underbrace{(g'' + \alpha n e^{-g} - \beta n e^{g})}_{=0}) + O(\Delta x^3)$$

A differential equation

- Drop terms of third order and higher
- Buffer sizing function must satisfy
  - $$f(x) = e^{g(x)}, \ x = \frac{1}{n+1} \ \forall i = 0,...,n+1$$
  - g(x) is a solution to the second-order ODE

$$g'' = \beta n e^{g} - \alpha n e^{-g}$$

with the boundary conditions

$$g(0) = \ln(\lambda), \ g(1) = \ln(\mu)$$

The ODE for buffer sizing function

- Apply chain rule of differentiation
- Undo previous substitution, e.g.

$$g(x) = \ln f(x)$$

- Buffer sizing function is the solution to the following first order ODE with given boundary conditions

$$f'^2 = 2n\beta f^3 + cf^2 + 2n\alpha f$$

- **C** can be determined numerically