An Efficient and Globally Convergent Homotopy Method for Finding DC Operating Points of Nonlinear Circuits

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Finding DC Operating Points of Nonlinear Circuits

The Newton-Raphson method employed in SPICE-like simulators often fails to converge to a solution. To overcome this convergence problem, homotopy methods have been studied from various viewpoints.

There are several types of homotopy methods, one of which succeeded in solving bipolar analog circuits with more than 20000 elements with the theoretical guarantee of global convergence [1]ー[5].

In this presentation, we propose an improved version of the homotopy method that can find DC operating points of practical nonlinear circuits smoothly and efficiently.

Homotopy Method

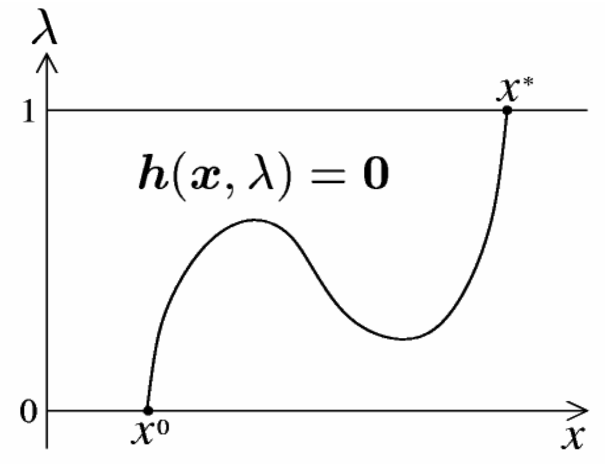
Modified nodal equation $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$ used in SPICE

Auxiliary equation with a known solution $\boldsymbol{x}^0$ $\quad \boldsymbol{f}^0(\boldsymbol{x}) = \mathbf{0}$

Define a homotopy function:

$$\boldsymbol{h}(\boldsymbol{x}, \lambda) = \lambda \boldsymbol{f}(\boldsymbol{x}) + (1 - \lambda)\boldsymbol{f}^0(\boldsymbol{x})$$

Then, the solution curve of the homotopy equation $\boldsymbol{h}(\boldsymbol{x}, \lambda) = \mathbf{0}$ is traced from the initial point $(\boldsymbol{x}^0, 0)$.



Computational Efficiency of Homotopy Methods

In the homotopy methods, the computational efficiency depends on the homotopy function (as well as the path following algorithm).

- Newton homotopy (NH)

$$\boldsymbol{h}(\boldsymbol{x},\lambda) = \boldsymbol{f}(\boldsymbol{x}) - (1-\lambda)\boldsymbol{f}(\boldsymbol{x}^0)$$

- Newton-fixed-point homotopy (NFPH)

$$\boldsymbol{h}(\boldsymbol{x},\lambda) = \boldsymbol{f}(\boldsymbol{x}) - (1-\lambda)\boldsymbol{f}(\boldsymbol{x}^0) + (1-\lambda)\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{x}^0)$$

- Variable gain homotopy (VGH)

$$\boldsymbol{h}(\boldsymbol{x},\lambda) = \boldsymbol{f}(\boldsymbol{x},\lambda\boldsymbol{\alpha}) + (1-\lambda)\boldsymbol{G}(\boldsymbol{x}-\boldsymbol{a})$$

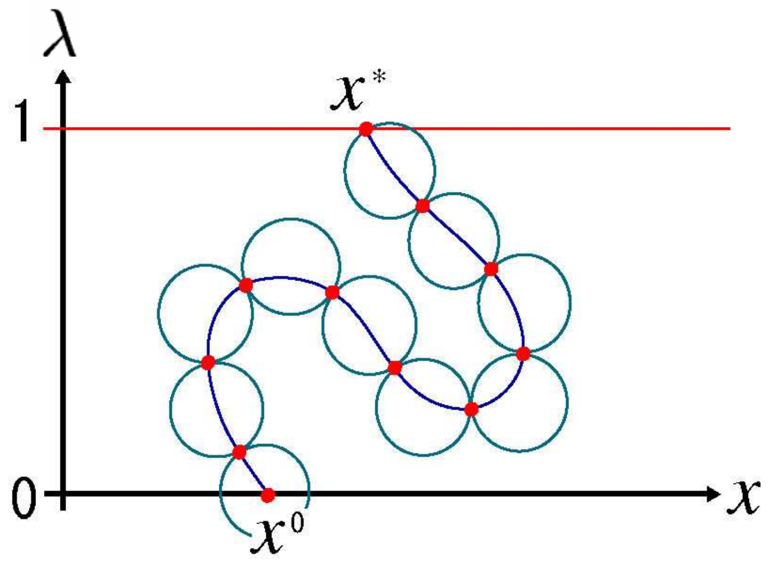
1. Newton Homotopy (NH) Method

$$\boldsymbol{h}(\boldsymbol{x},\lambda)=\boldsymbol{f}(\boldsymbol{x})-(1-\lambda)\boldsymbol{f}(\boldsymbol{x}^0)$$

The homotopy method using the Newton homotopy and the path following algorithm using hyperspheres [1],[2] succeeded in solving bipolar analog circuits with more than 20000 elements (that belong to a class of the largest-scale circuits available with the current bipolar analog LSI technology) with the theoretical guarantee of global convergence [1]–[5].

However, the global convergence of the NH method is guaranteed only when we choose an initial point on which the uniform passivity holds.

Hence, we cannot use *good initial points*, for example, points in the forward active operation region.



2. Newton-Fixed-Point Homotopy (NFPH) Method

$$\boldsymbol{h}(\boldsymbol{x},\lambda) = \boldsymbol{f}(\boldsymbol{x}) - (1-\lambda)\boldsymbol{f}(\boldsymbol{x}^0) + \underbrace{(1-\lambda)\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{x}^0)}_{\text{linear term}}$$

- The Newton-fixed-point homotopy (NFPH) method is an improved version of the NH method [6],[7].
- In this method, we can trace a solution curve from a good initial point, which often makes the solution curve short and makes the algorithm efficient.
- However, this homotopy contains a linear function that has no relation to the original nonlinear function, which sometimes causes complicated movement of solution curves.

3. Variable-Gain Homotopy (VGH) Method

$$h(x, \lambda) = f(x, \lambda\alpha) + (1 - \lambda)G(x - a)$$

- As another efficient approach of the homotopy method, using the variable-gain homotopy (VGH) is well-known [8], where α is a vector consisting of forward and reverse current gains of transistors.
- Since this method includes the excellent idea of variable gain, solution curves often become smooth.
- However, in this method, we sometimes have to trace a solution curve from an initial point far from the solution; namely, in this method, we cannot choose x^0 because it is obtained by solving a special diode circuit.

⟹ More efficient homotopy method

Purpose of this Study

In this presentation, we propose an efficient homotopy method (VGNH method) that is based on the idea of the NFPH method and that of the VGH method.

- The auxiliary equation at $\lambda = 0$ is closely related to the original nonlinear equation.
- Since this method is globally convergent for any initial point, we can choose a good initial point.
- The idea of variable gain is introduced.

Therefore, we can trace solution curves smoothly and efficiently.

The proposed method can be easily implemented on SPICE without programming.

Modified nodal equation $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$

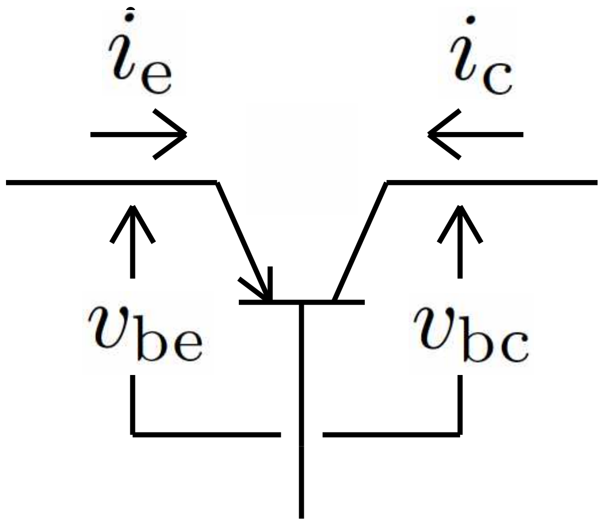
$$\begin{aligned} \boldsymbol{f}_g(\boldsymbol{v},\boldsymbol{i}) &\triangleq \boldsymbol{D}_g \boldsymbol{g}(\boldsymbol{D}_g{}^T \boldsymbol{v}) + \boldsymbol{D}_E \boldsymbol{i} + \boldsymbol{J} = \boldsymbol{0} \\ \boldsymbol{f}_E(\boldsymbol{v},\boldsymbol{i}) &\triangleq \boldsymbol{D}_E{}^T \boldsymbol{v} - \boldsymbol{E} = \boldsymbol{0}, \end{aligned}$$

For simplicity, assume that bipolar junction transistors are described by the Ebers-Moll model:

$$\boldsymbol{v}_q = (v_{\mathrm{be}}, v_{\mathrm{bc}})^T \qquad \boldsymbol{i}_q = (i_{\mathrm{e}}, i_{\mathrm{c}})^T$$

$$\boldsymbol{i}_q(\boldsymbol{v}_q) = \boldsymbol{T}\boldsymbol{q}(\boldsymbol{v}_q), \qquad \boldsymbol{T} = \begin{bmatrix} 1 & -\alpha_r \\ -\alpha_f & 1 \end{bmatrix}$$

$$\boldsymbol{q}(\boldsymbol{v}_q) = \begin{bmatrix} m_e(\exp(n_e v_{\mathrm{be}}) - 1) \\ m_c(\exp(n_c v_{\mathrm{bc}}) - 1) \end{bmatrix}$$



Proposed Method

- In the proposed method, we use

$$\boxed{\boldsymbol{h}(\boldsymbol{x},\lambda) = \boldsymbol{f}(\boldsymbol{x},\lambda\boldsymbol{\alpha}) - (1-\lambda)\boldsymbol{f}(\boldsymbol{x}^0, 0\cdot\boldsymbol{\alpha})}$$

- If we consider a circuit described by $\boldsymbol{h}(\boldsymbol{x},\lambda) = \boldsymbol{0}$, then each transistor of the circuit can be described by

$$\boldsymbol{i}_q(\boldsymbol{v}_q) = \boldsymbol{T}\boldsymbol{q}(\boldsymbol{v}_q)$$

with T represented by

$$\boldsymbol{T}_\lambda = \begin{bmatrix} 1 & -\lambda\alpha_r \\ -\lambda\alpha_f & 1 \end{bmatrix}$$

If $\lambda = 0$, then the corresponding circuit contains diodes as only nonlinear elements, hence it has a unique solution.

Global Convergence Property

For the global convergence property of the proposed method, the following theorem holds.

Theorem 1: Assume that $\boldsymbol{g}$ is uniformly passive [4] on certain points. Then, for any initial point $\boldsymbol{x}^0$, the solution curve of $\boldsymbol{h}(\boldsymbol{x}, \lambda) = \boldsymbol{0}$ starting from $(\boldsymbol{x}^0, 0)$ reaches $\lambda = 1$.

Thus, the proposed method is proven to be globally convergent for MN equations from any initial point.

Computational Efficiency

Next, we discuss the computational efficiency of the proposed method, considering the factors that degrade the efficiency in the conventional methods stated before. Namely, we show that the proposed method is free from the difficulties of the VGH method and the NFPH method.

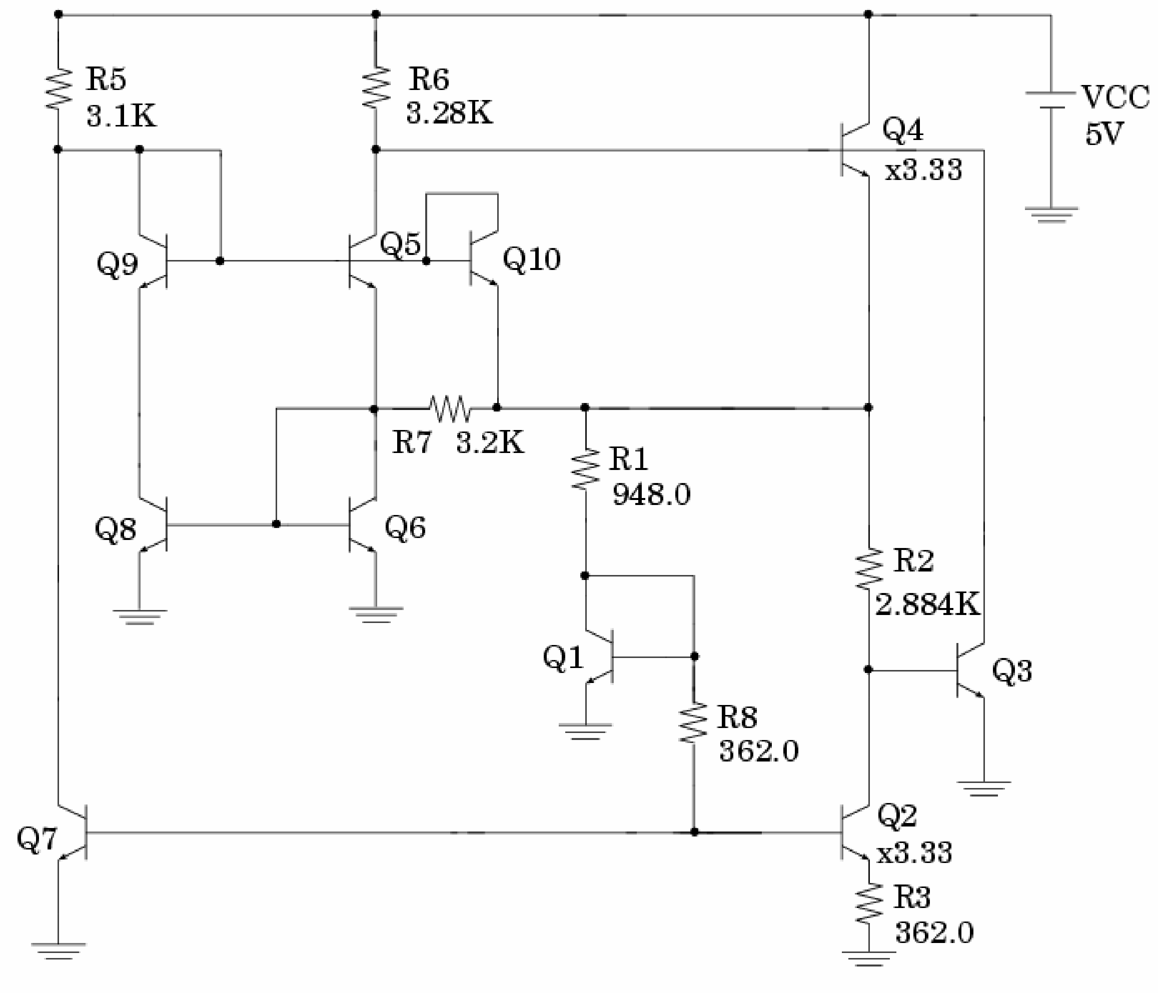
- First, since $\boldsymbol{h}(\boldsymbol{x}^0, 0) = \mathbf{0}$ holds for any $\boldsymbol{x}^0$, we can choose a good initial point (unlike the VGH method).
- Secondly, the proposed homotopy function contains no linear function, and the auxiliary equation $\boldsymbol{h}(\boldsymbol{x}, 0) = \mathbf{0}$ is closely related to the original nonlinear equation $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$. Hence, the proposed method is free from the problem of the NFPH method.
- Moreover, since the proposed method includes the concept of variable gain, it is expected that solution curves become smooth and short.

Numerical Examples

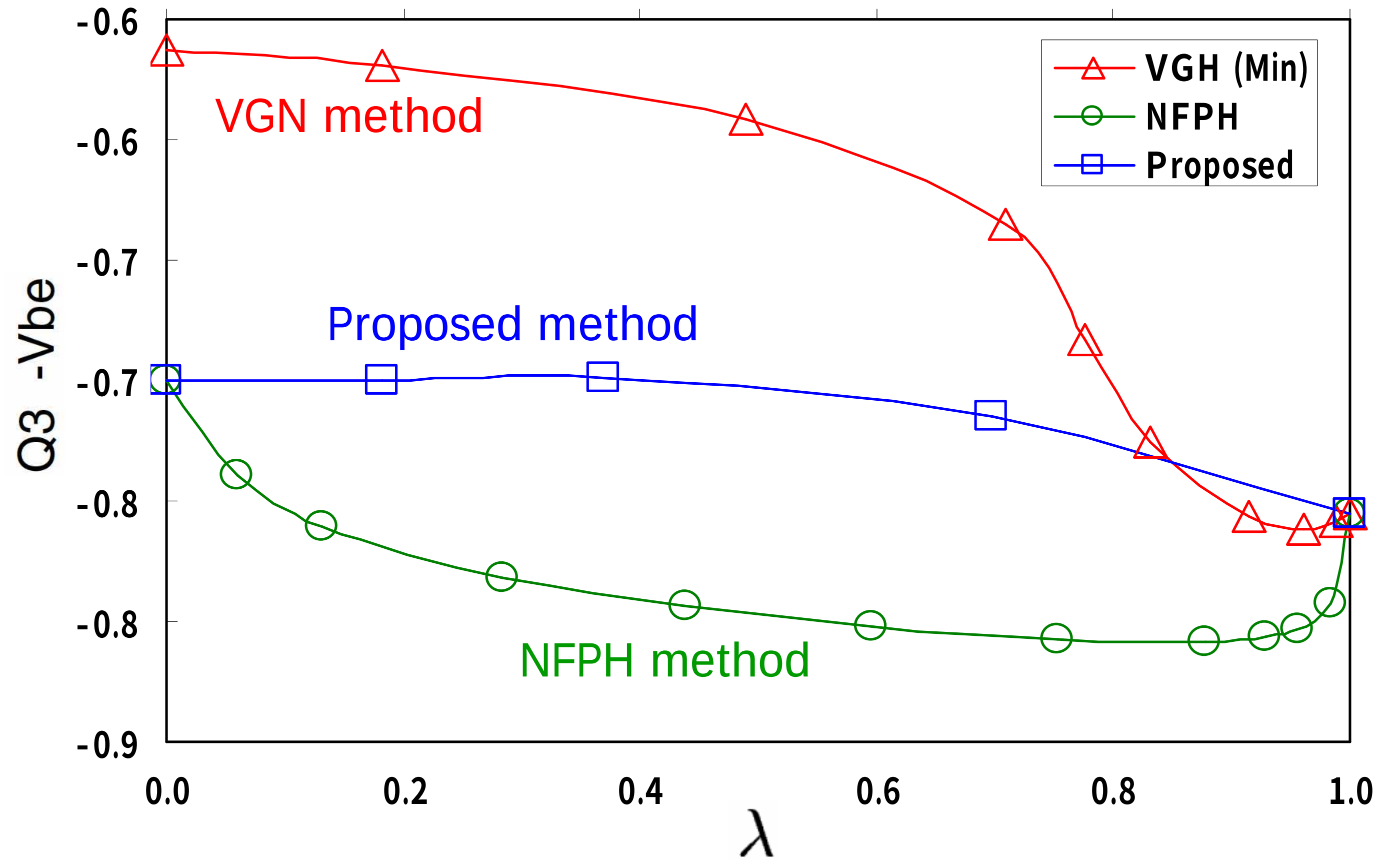
We implemented the proposed method on a Sun Blade 2000 and have confirmed the effectiveness of the proposed method using many practical transistor circuits. In all of the numerical experiments we have performed, the proposed method was the most efficient. We show some computational results, where we chose the initial points in the forward active operation region for all transistors ($\boldsymbol{v}_q^0 = (0.7, 0)^T$) and used the spherical method [2] for tracing solution curves.

[2] K. Yamamura, "Simple algorithms for tracing solution curves," IEEE Trans. Circuits & Syst.-I, vol.40, no.8, pp.537–541, Aug. 1993.

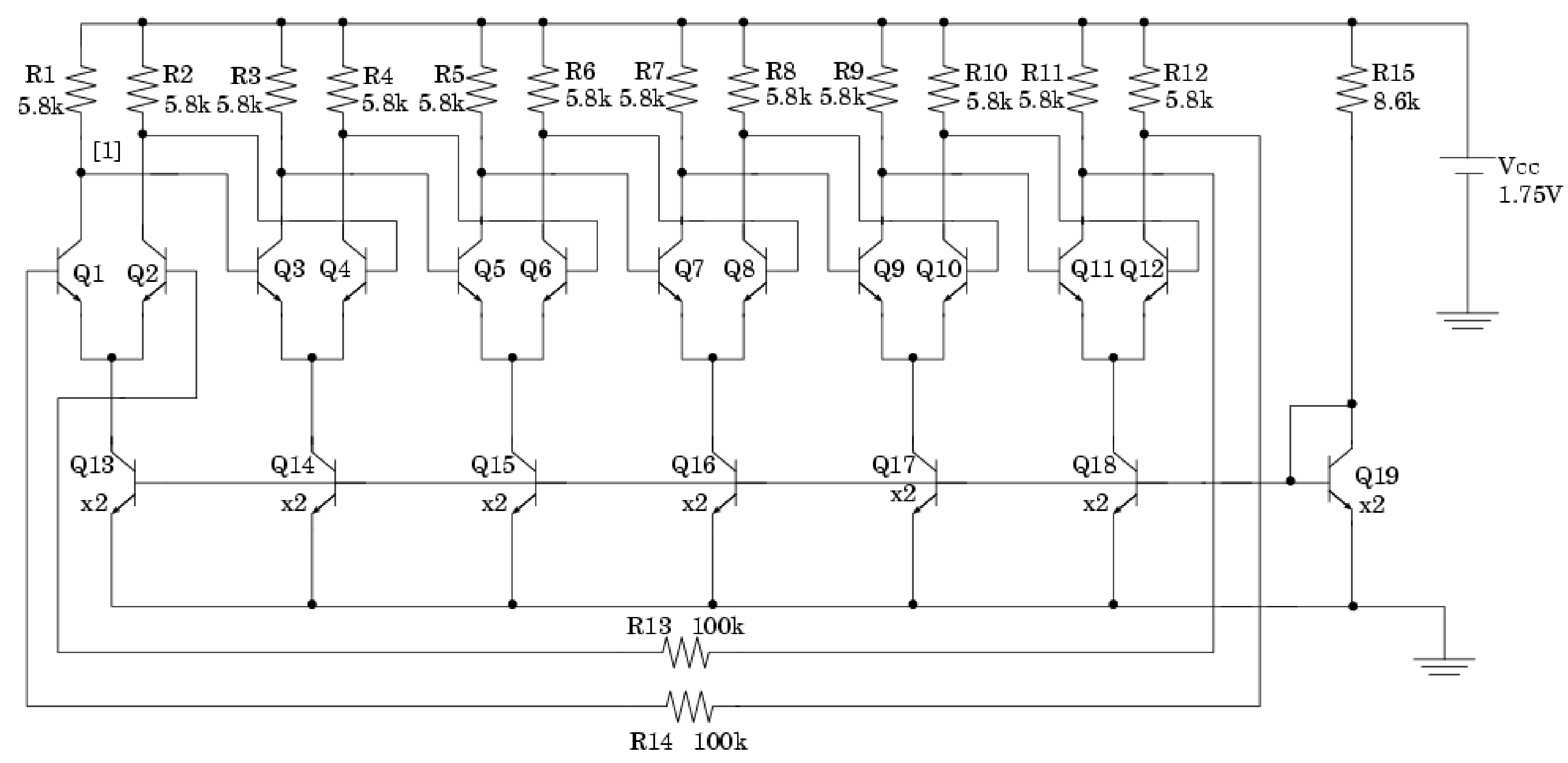
Hybrid Voltage Reference Circuit



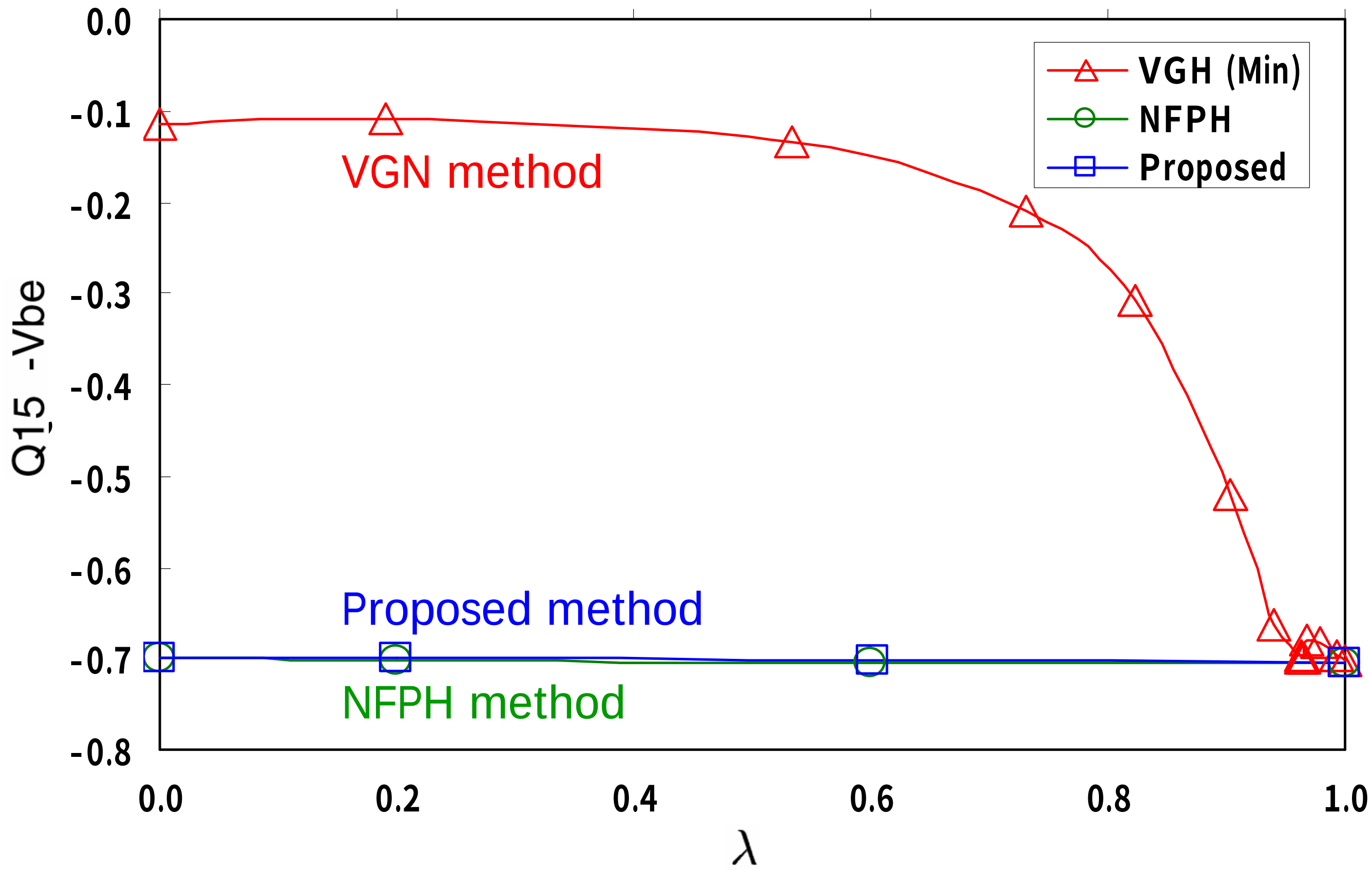
In the figure, the emitter-to-base voltage v_{be} of a certain BJT is plotted, where marks indicate the steps. In each step, a system of $n+1$ nonlinear equations is solved by the NR method. From this figure, it is seen that the proposed method traces solution curves more smoothly and efficiently than the conventional methods (VGH and NFPH).



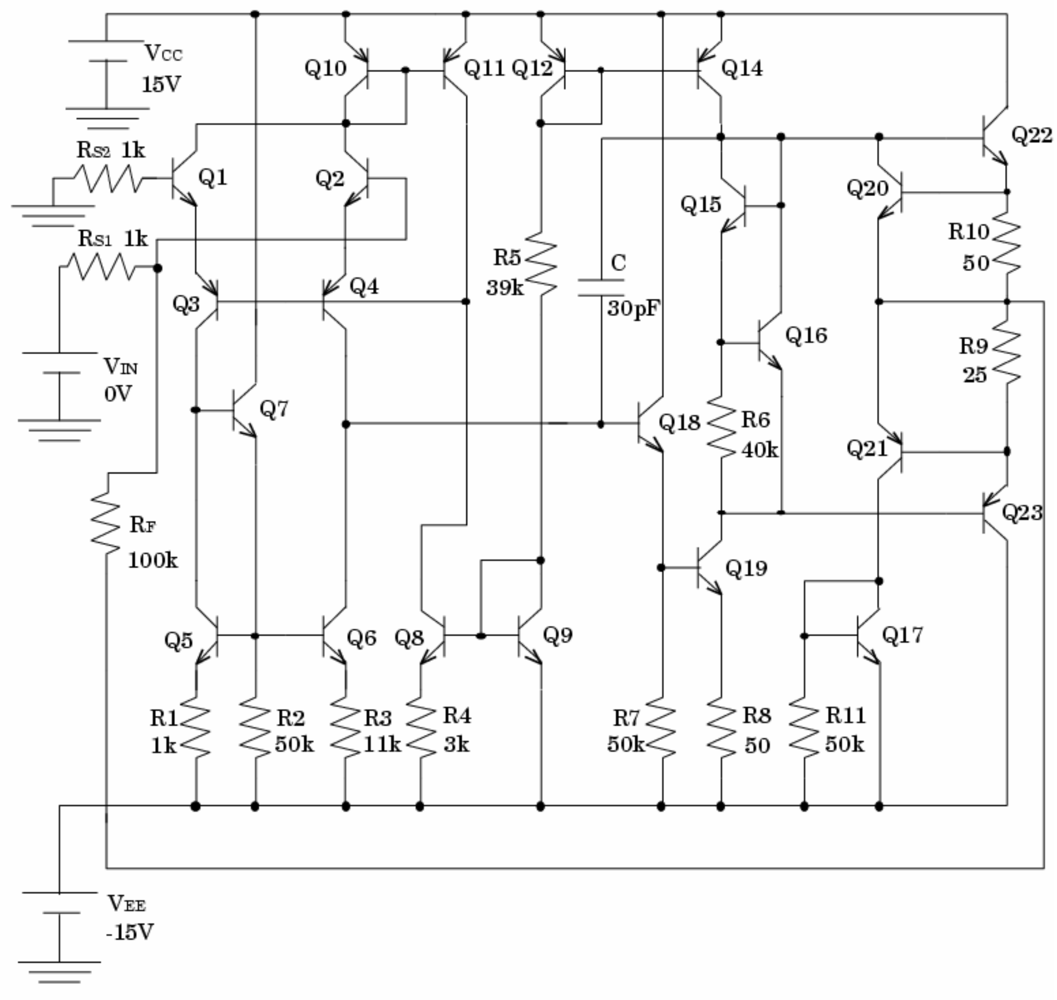
Six-Stage Limiting Amplifier



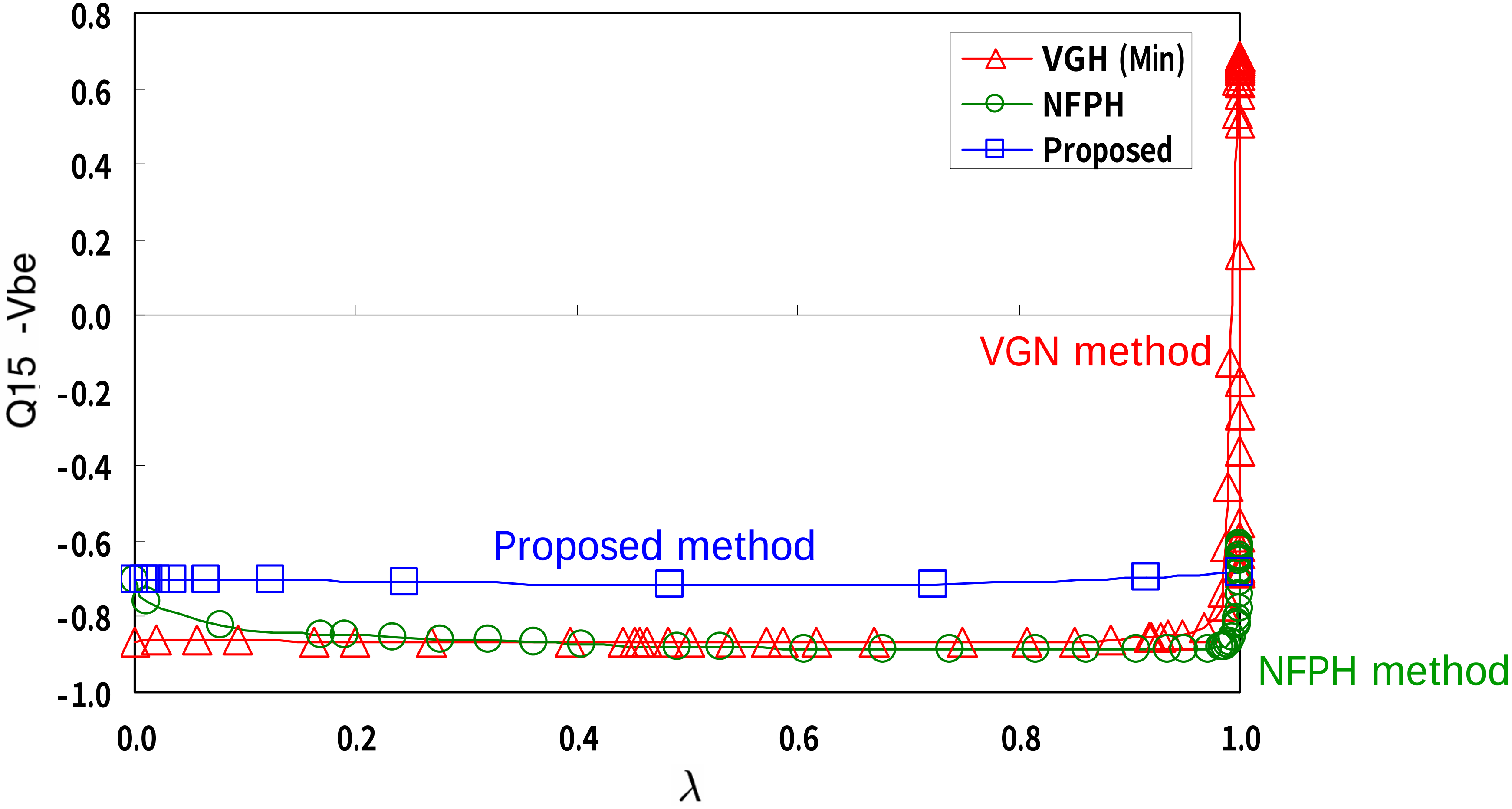
Six-Stage Limiting Amplifier



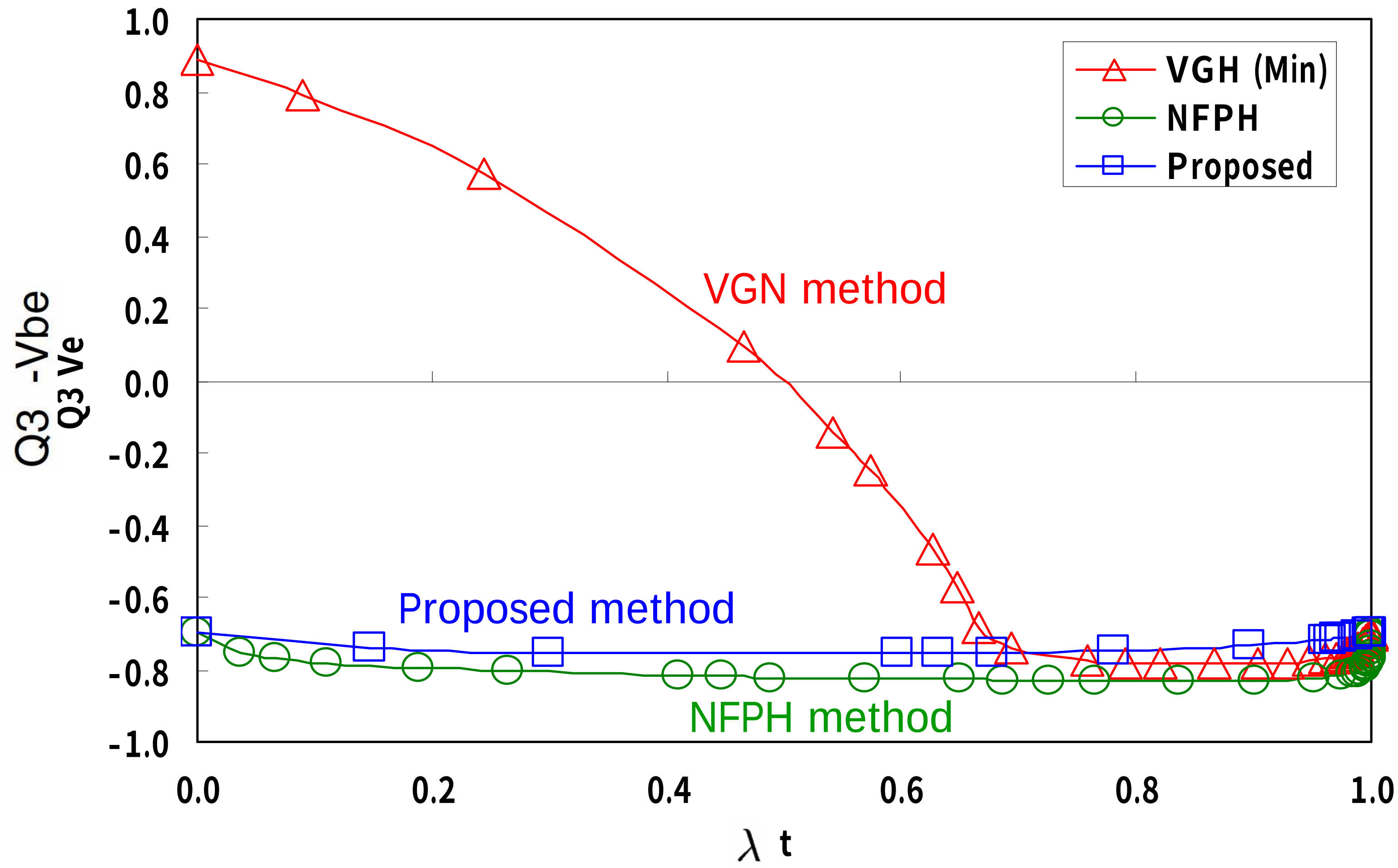
High-Gain Operational Amplifier μA741



High-Gain Operational Amplifier μA741

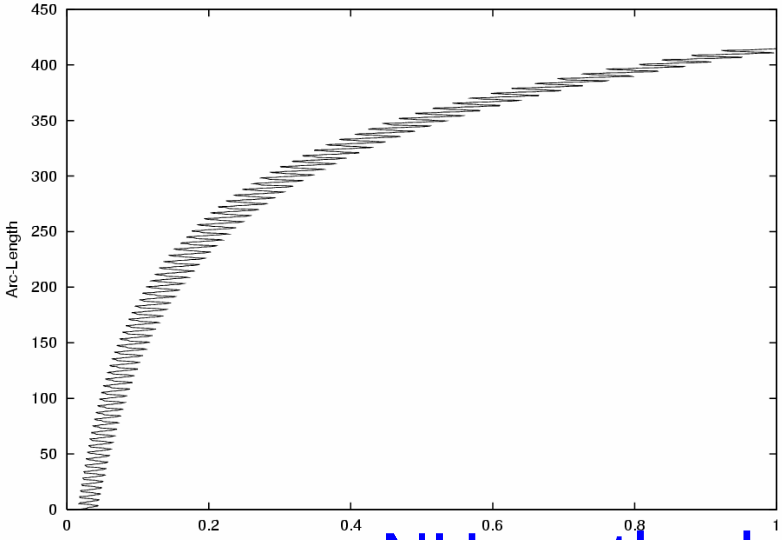


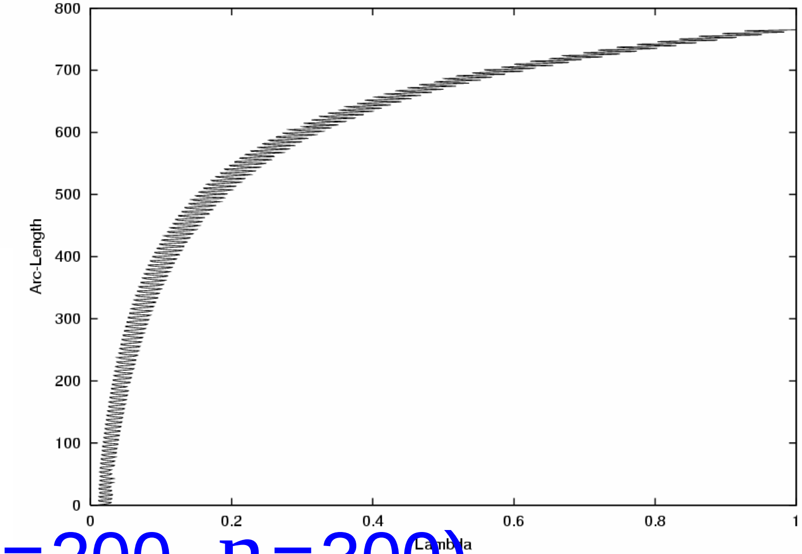
Regulator Circuit (41 elements, 24 BJT)



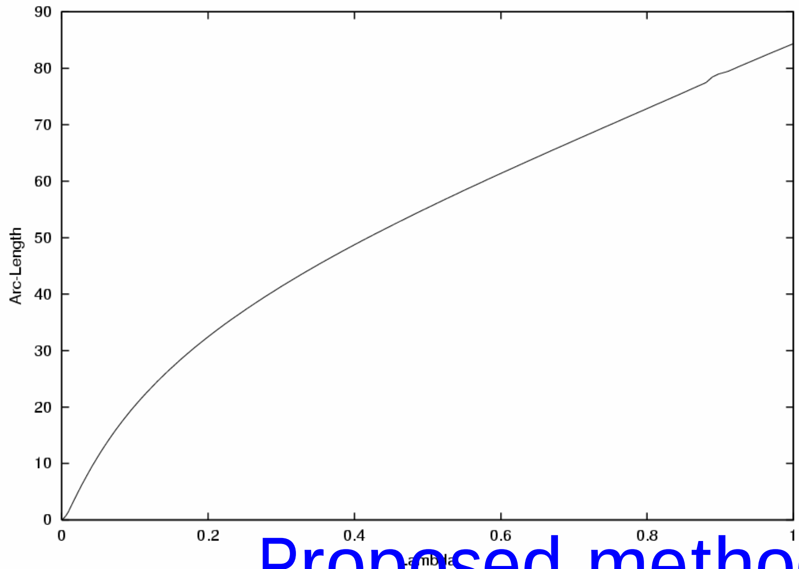
Circuits containing n tunnel diodes

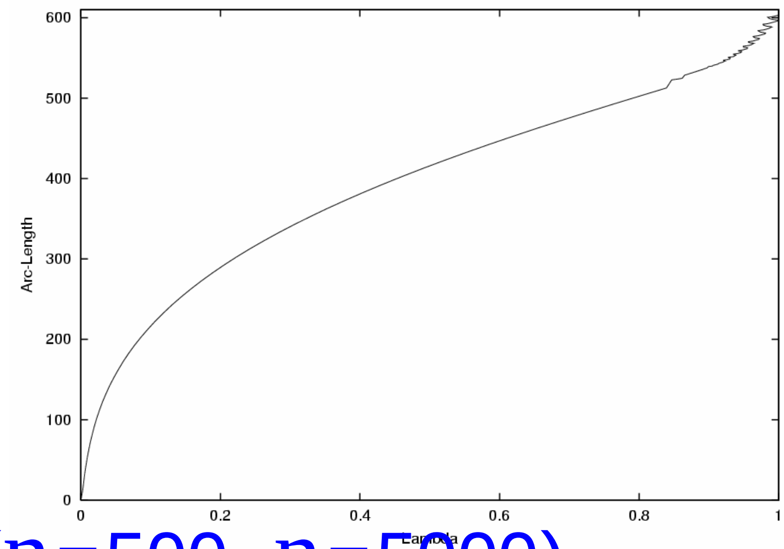
$$g(v,\lambda) = av^3 - \underline{\lambda} bv^2 + cv, \qquad a,b,c > 0$$





NH method (n=200, n=300)





Proposed method (n=500, n=5000)

Convergence rate (%)

when we applied the two methods from randomely chosen one hundred initial points.

(The global convergence of the NH method is guaranteed only when we choose an initial point on which the uniform passivity hold [1],[4])

n	NH	Proposed
10	29 (%)	100 (%)
20	15	100
50	1	100
100	0	100

Comparison of computation time T (s) $(\boldsymbol{x}^0 = 0)$

	NH method			Proposed method		
n	Steps	Length	T (s)	Steps	Length	T (s)
500	5 040	1 570	87	22	83	0.5
1000	16 040	4 480	1 591	52	153	8
1500	57 838	8 026	15 037	88	215	51
2000	83 941	12 352	55 410 15h	120	282	190 3m
2500	–	–	–	151	339	416
3000	–	–	–	145	389	658
3500	–	–	–	166	443	1 059
4000	–	–	–	176	495	2 977
4500	–	–	–	251	544	3 278
5000	–	–	–	236	600	3 934 1h

Implementation of the VGNH Method on SPICE without Programming

- Thus, the proposed method is not only globally convergent for any initial point but also efficient because we can use good initial points and the solution curves tend to become smooth and short. In a sense, the proposed method has all the advantages of the NH, NFPH, and VGH methods, and is free from the difficulties of these methods.
- <u>However, the programming of sophisticated homotopy methods is often difficult for non-experts or beginners.</u>
- In this presentation, we also propose an effective method for implementing the VGNH method on SPICE using the idea of the SPICE-oriented numerical methods [1],[11]ー[13].

We first note that the VGNH, namely,

$$\boxed{\boldsymbol{h}(\boldsymbol{x},\lambda) = \boldsymbol{f}(\boldsymbol{x},\lambda\boldsymbol{\alpha}) - (1-\lambda)\boldsymbol{f}(\boldsymbol{x}^0, 0\cdot\boldsymbol{\alpha})}$$

$\Downarrow$ is equivalent to

$$\boxed{\boldsymbol{h}(\boldsymbol{x},\lambda) = \boldsymbol{f}(\boldsymbol{x}) + (1-\lambda)\tilde{\boldsymbol{f}}(\boldsymbol{x}) - (1-\lambda)(\boldsymbol{f}(\boldsymbol{x}^0) + \tilde{\boldsymbol{f}}(\boldsymbol{x}^0))}$$

where

$$\tilde{\boldsymbol{f}}(\boldsymbol{x}) \triangleq \begin{bmatrix} \boldsymbol{D}_g\tilde{\boldsymbol{g}}(\boldsymbol{D}_g^T\boldsymbol{v}) \\ \boldsymbol{0} \end{bmatrix}, \quad \begin{bmatrix} \tilde{g}_i \\ \tilde{g}_{i+1} \end{bmatrix} = \tilde{\boldsymbol{T}}\boldsymbol{q}(\boldsymbol{v}_q), \quad \tilde{\boldsymbol{T}} = \begin{bmatrix} 0 & \alpha_r \\ \alpha_f & 0 \end{bmatrix}$$

As for the proof, please see the Proceedings.

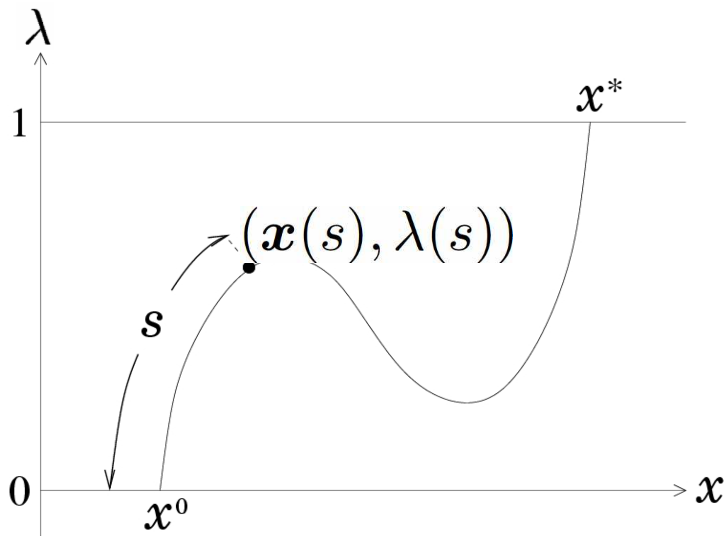
Tracing Solution Curves

In the VGNH method, the solution curve can be traced by integrating a system of algebraic-differential equations:

$$\boldsymbol{f}(\boldsymbol{x}) + (1-\lambda)\tilde{\boldsymbol{f}}(\boldsymbol{x}) - (1-\lambda)\left(\boldsymbol{f}(\boldsymbol{x}^0) + \tilde{\boldsymbol{f}}(\boldsymbol{x}^0)\right) = \boldsymbol{0} \qquad (25)$$

$$\sum_{i=1}^{m}\left(\frac{dv_{\mathrm{be}i}}{ds}\right)^2 + \left(\frac{d\lambda}{ds}\right)^2 = 1$$

starting from $(\boldsymbol{x}^0, 0)$ [11]–[13], where s denotes the arc-length of the solution curve. Note that the points on the solution curve are considered as functions of s.



SPICE-Oriented Approach

- In the approach of the SPICE-oriented numerical methods, we consider a circuit described by (25). Then, we perform the transient analysis of SPICE to the circuit starting from $(\boldsymbol{x}^0, 0)$, by which numerical integration is applied to (25) and the solution curve of $\boldsymbol{h}(\boldsymbol{x}, \lambda) = \mathbf{0}$ is traced.
- Now we consider to describe

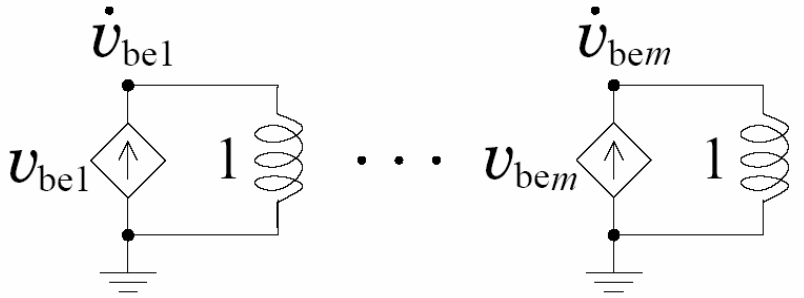
$$\boldsymbol{f}(\boldsymbol{x}) + (1-\lambda)\tilde{\boldsymbol{f}}(\boldsymbol{x}) - (1-\lambda)(\boldsymbol{f}(\boldsymbol{x}^0) + \tilde{\boldsymbol{f}}(\boldsymbol{x}^0)) = \mathbf{0} \quad \text{(25a)}$$

$$\sum_{i=1}^{m} \left(\frac{dv_{\mathrm{bei}}}{ds}\right)^2 + \left(\frac{d\lambda}{ds}\right)^2 = 1 \quad \text{(25b)}$$

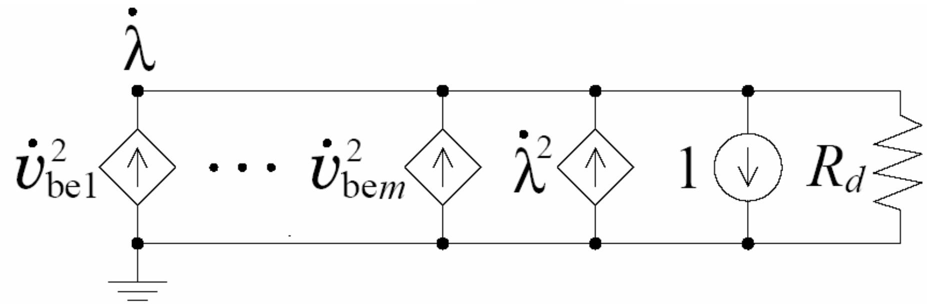
by circuits.

Description of (25b)

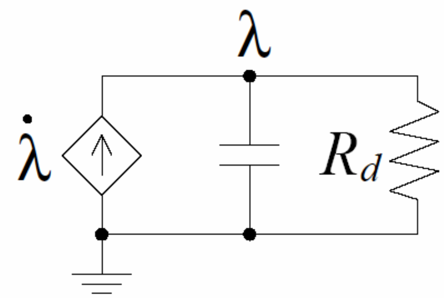
$$\sum_{i=1}^{m}\left(\frac{dv_{\mathrm{be}i}}{ds}\right)^2+\left(\frac{d\lambda}{ds}\right)^2=1$$ by circuits [11]–[13]



$$\dot{v}_{\mathrm{be}i}=\frac{dv_{\mathrm{be}i}}{ds}$$



$$\sum_{i=1}^{m}\left(\dot{v}_{\mathrm{be}i}\right)^2+\left(\dot{\lambda}\right)^2=1$$



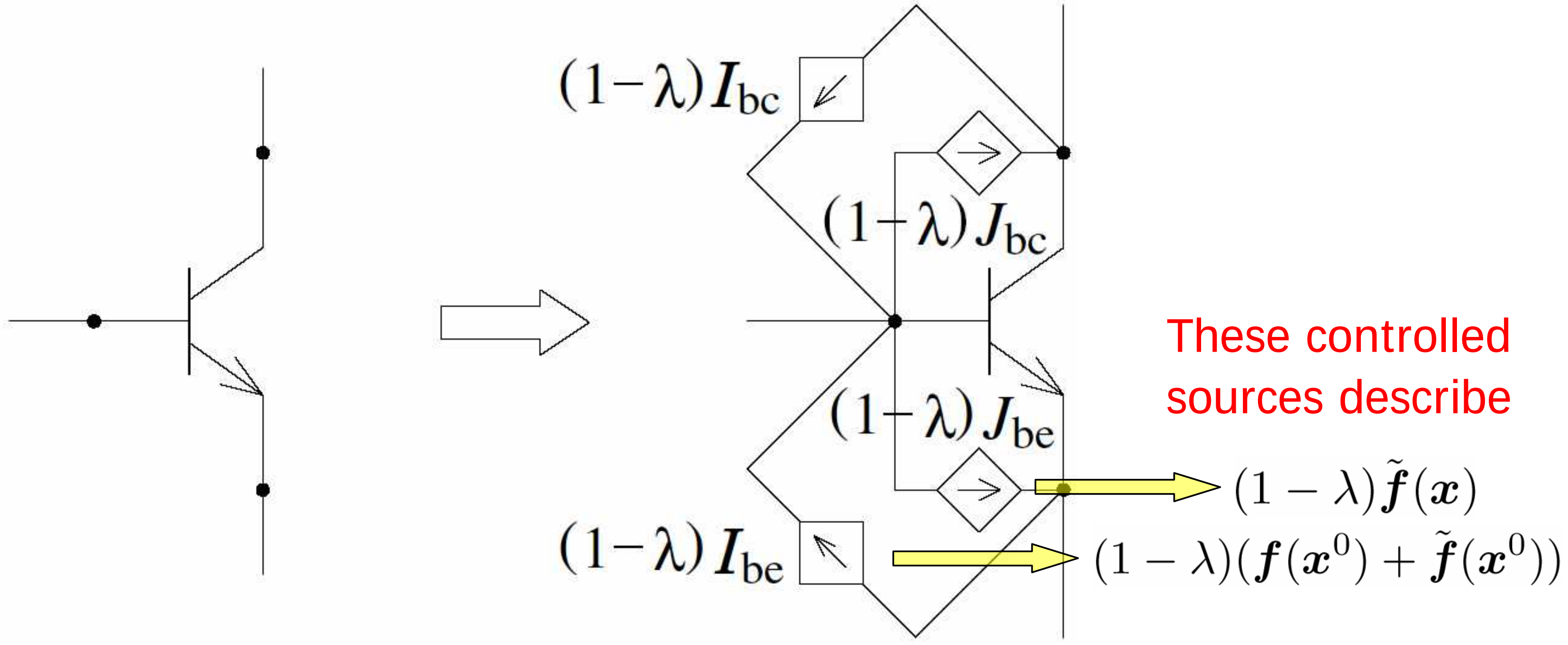
$$\dot{\lambda}=\frac{d\lambda}{ds}$$

Description of (25a)

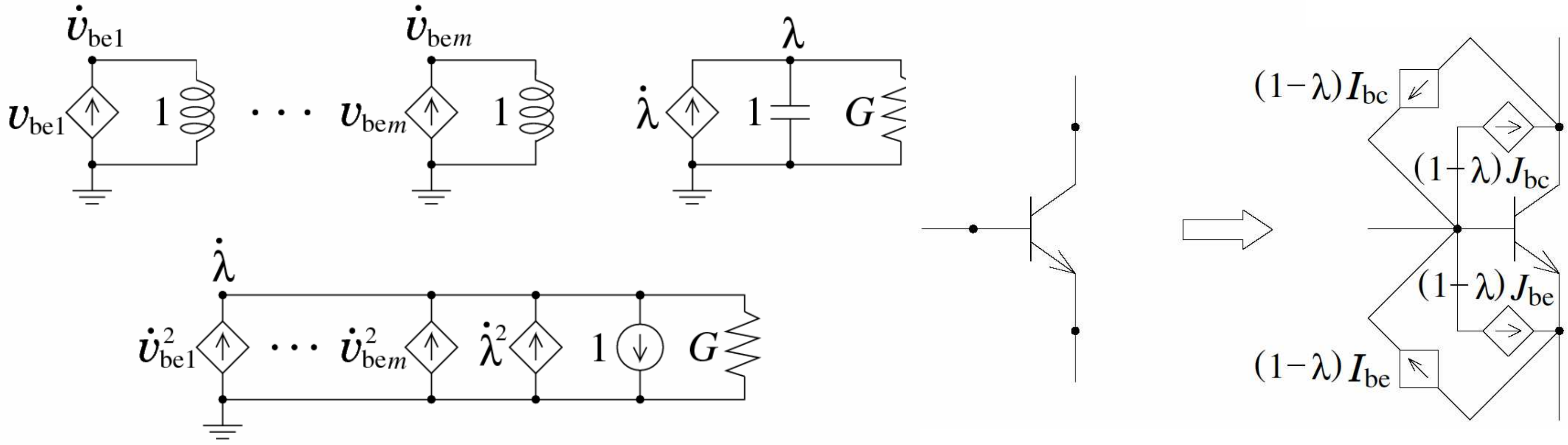
$$\boldsymbol{f}(\boldsymbol{x}) + (1-\lambda)\tilde{\boldsymbol{f}}(\boldsymbol{x}) - (1-\lambda)\left(\boldsymbol{f}(\boldsymbol{x}^0) + \tilde{\boldsymbol{f}}(\boldsymbol{x}^0)\right) = \boldsymbol{0}$$

by a circuit

Next, (25a) is described by a circuit like this, where four controlled current sources are connected to each transistor of the original circuit. (For details, see the Proceedings.)



By performing the transient analysis of SPICE to the circuits like



starting from $(\boldsymbol{x}^0, 0)$, we can trace the solution curve of $\boldsymbol{h}(\boldsymbol{x}, \lambda) = \boldsymbol{0}$ and obtain a solution $\boldsymbol{x}^*$ at $\lambda = 1$.

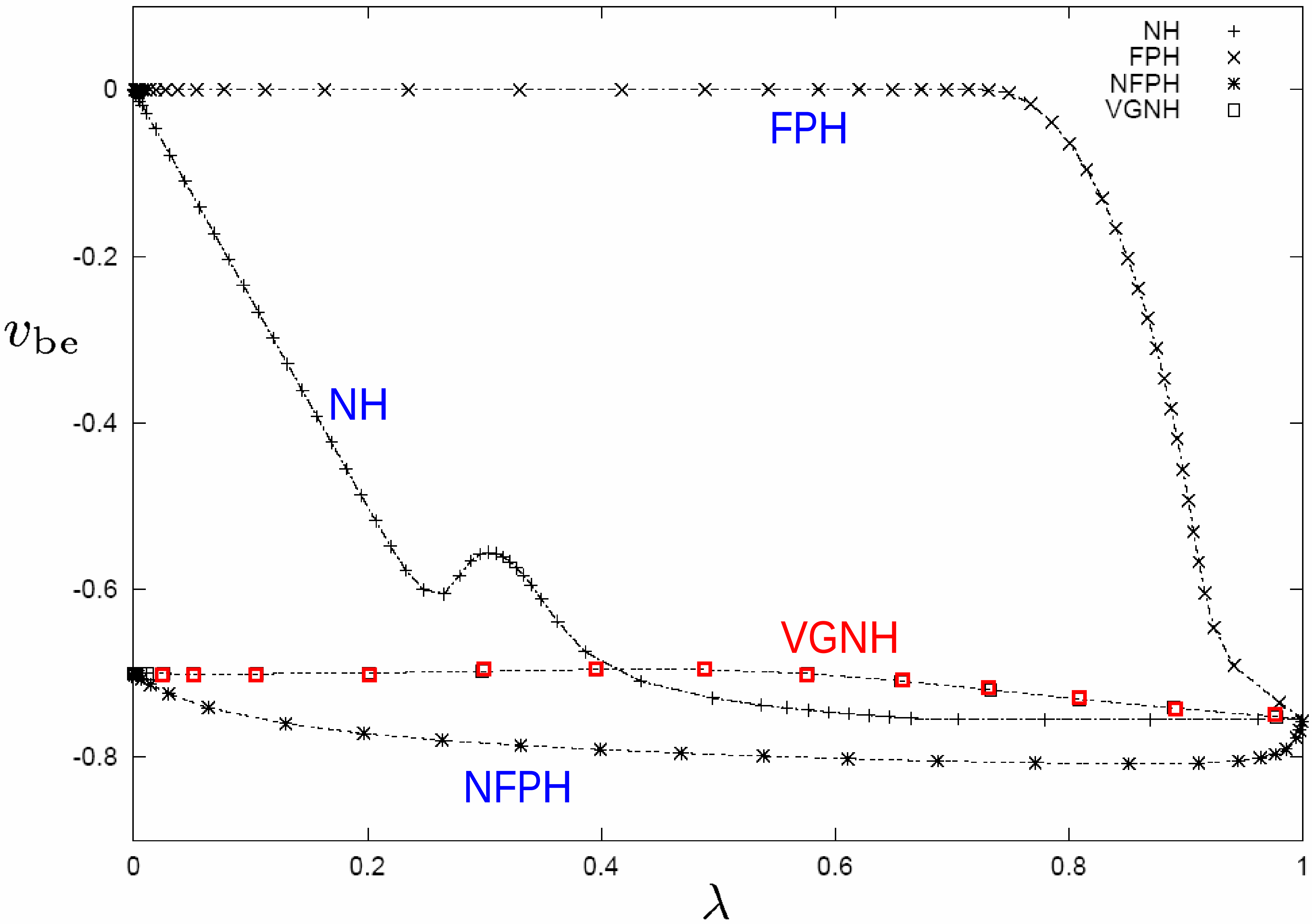
Since SPICE contains various efficient techniques such as sparse matrix techniques, implicit integration methods, and time-step control algorithms, a *high-level* VGNH method can be realized by this method. Moreover, *programming is not necessary,* and making the netlist of these circuits is quite easy.

Namely,

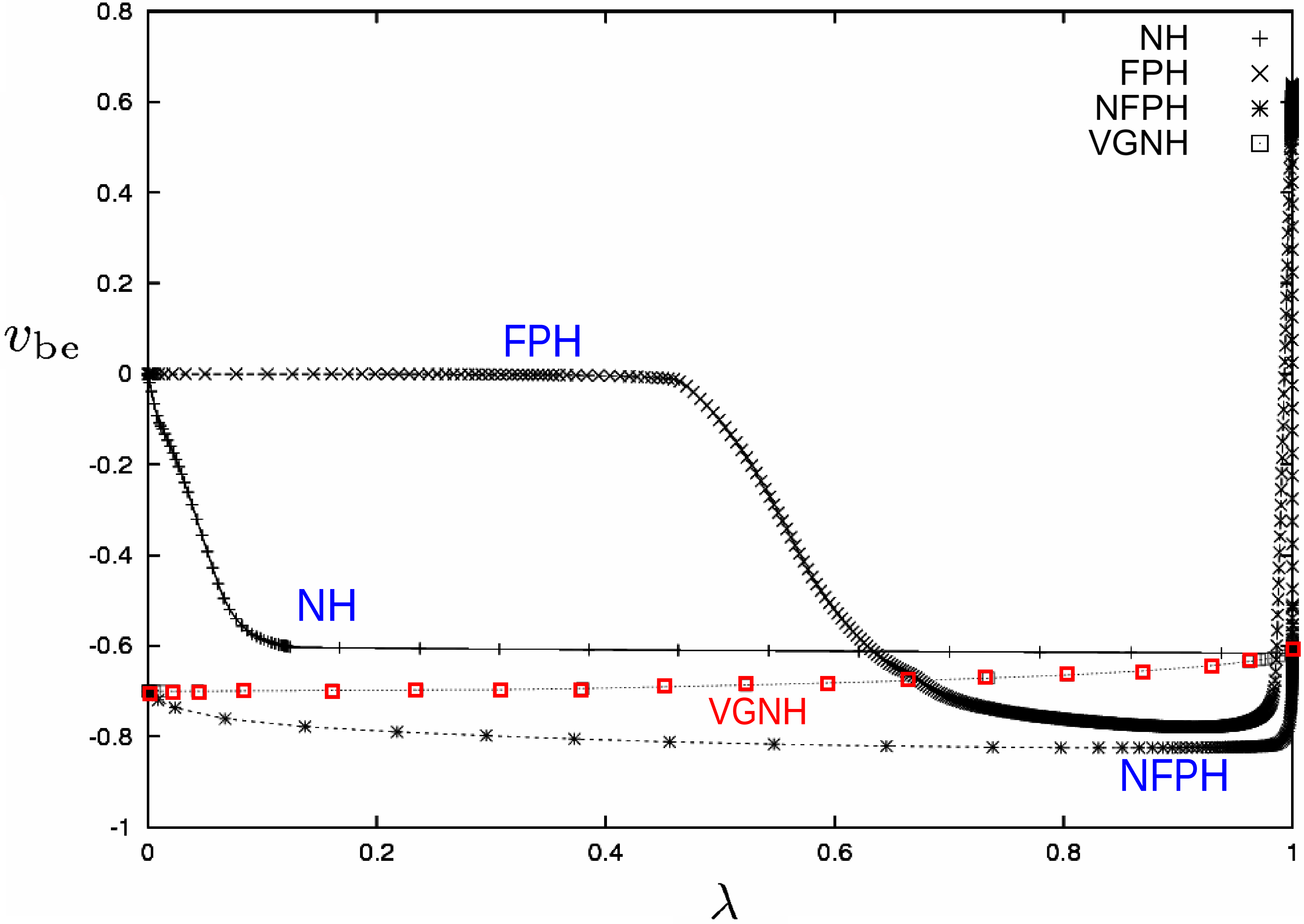
By this method, we can implement a

- sophisticated VGNH method with various efficient techniques
- easily
- without programming,
- although we do not know the homotopy method well.

Hybrid Voltage Reference Circuit



High-Gain Operational Amplifier μA741



The SPICE-oriented method was more efficient than using our own program as shown in this table. This is because SPICE contains various efficient techniques such as the sparse matrix techniques. Thus, we can implement an efficient VGNH method by using SPICE.

TABLE IV

COMPARISON OF COMPUTATION TIME.

Circuit	n	Program S	Program T (s)	SPICE S	SPICE T (s)
HVRef	41	21	0.117	21	0.060
2sOA	42	19	0.133	19	0.020
6sLA	80	19	0.500	19	0.120
μA741	95	28	1.517	28	0.320
RegCkt	95	48	2.600	48	0.400

Conclusion

- The proposed method is globally convergent for any initial point. (Theorem 1)
- The proposed method is very efficient.
- The proposed method can start from good initial points.
- The proposed method includes the concept of variable gain, but does not include linear functions.
- The proposed method can be easily implemented on SPICE without programming.