Balanced Truncation for Time-Delay Systems Via Approximate Gramians

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Outline

- Introduction and Objectives
- Problem Formulation
- Proposed Algorithm
- Experimental Results
- Summary and Conclusions

Introduction:
Model Order Reduction (MOR)

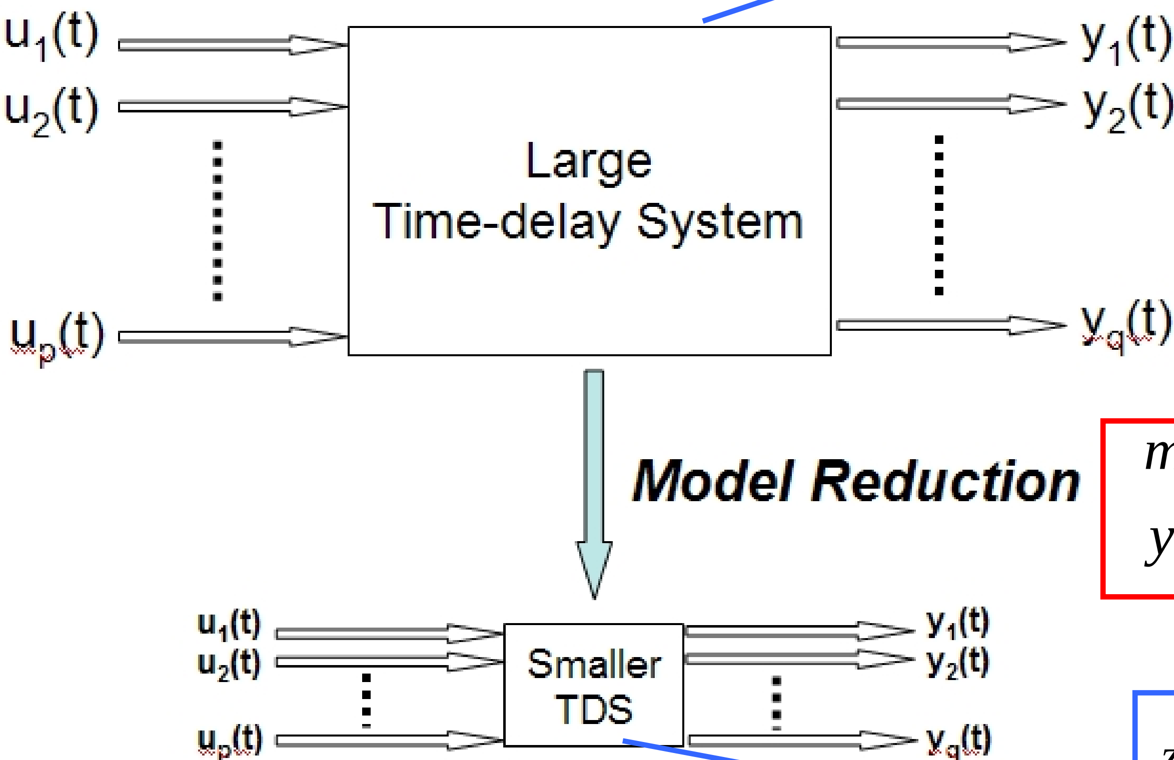
- Exponentially increasing elements are required in the original model for VLSI circuit simulation.

- MOR techniques compact the large model into a reduced-order model.

- Motivation is to reduce the internal complexity while preserving external behaviors.

Introduction:
Model Order Reduction

$$\dot{x}(t) = Ax(t) + A_d x(t-d) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x \in \Re^n$$



$$m << n$$
$$y_r \approx y$$

$$\dot{z}(t) = A_r z(t) + A_{dr} z(t-d) + B_r u(t)$$
$$y_r(t) = C_r z(t) + D_r u(t) \qquad z \in \Re^m$$

Introduction:
Time-Delay Systems (TDSs)

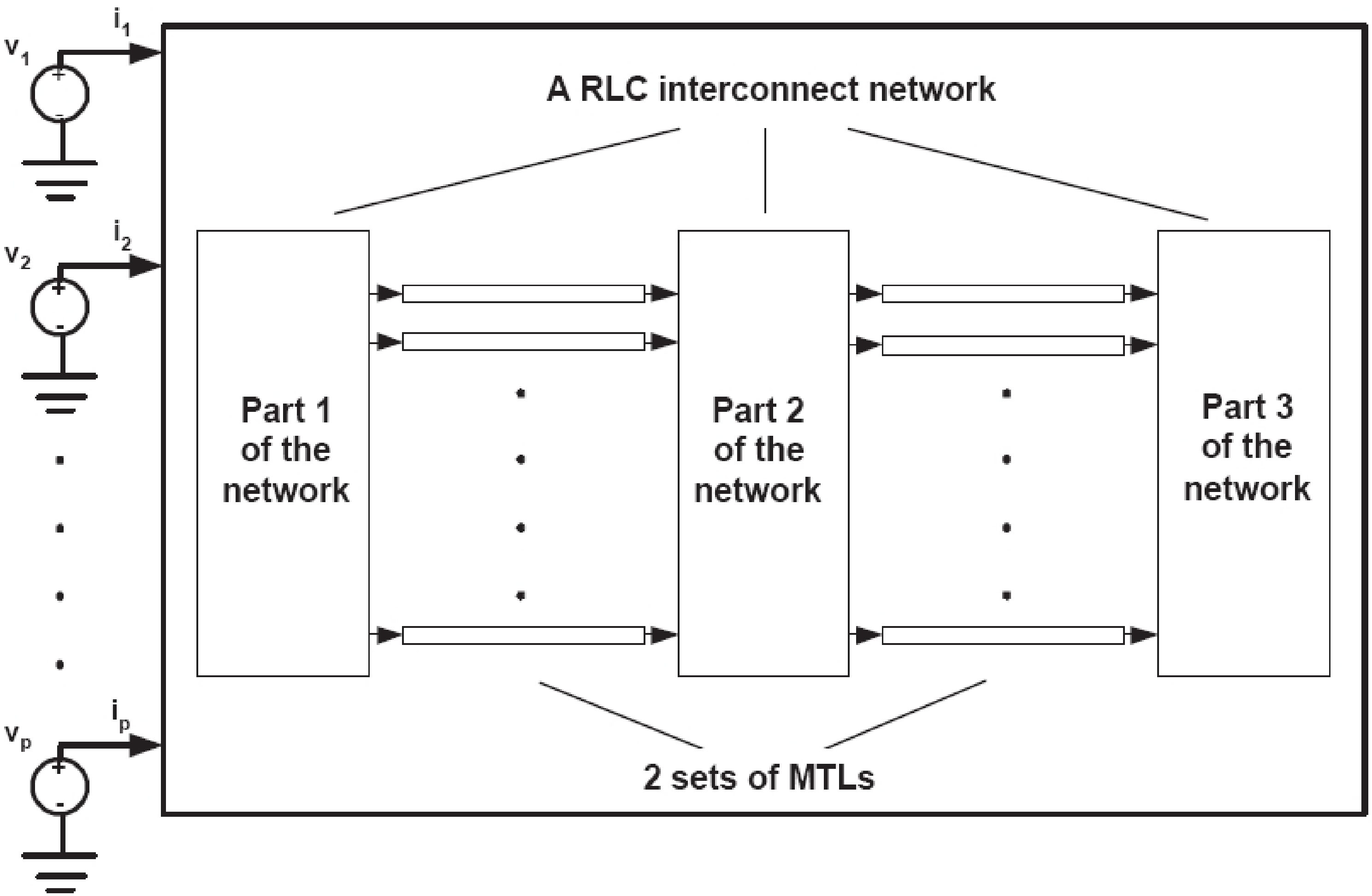
- A TDS may arise from a circuit network connected with delay elements such as transmission lines.

- MOR can be performed to compact the model for simulation efficiency.

- It’s the first time to reduce a TDS utilizing the balanced truncation approach.

- It provides a better accuracy than conventional approaches.

Introduction:
Time-Delay Systems (TDSs)



Problem Formulation:
Model Reduction of Time-Delay Systems

Original Large TDS:

$$\dot{x}(t) = Ax(t) + A_d x(t-d) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x \in \Re^n$$

Reduction by Projection

$$A_r = V^T A V, A_{dr} = V^T A_d V$$
$$B_r = V^T B, C_r = CV, D_r = D$$

$$m << n$$
$$y_r \approx y$$

$$x = V z$$

- The key is how to construct the projection matrix V for a time-delay system.

- The difficulty is how to deal with the delay term.

Reduced Small TDS:

$$\dot{z}(t) = A_r z(t) + A_{dr} z(t-d) + B_r u(t)$$
$$y_r(t) = C_r z(t) + D_r u(t) \qquad z \in \Re^m$$

Problem Formulation:
Taylor Expansion based MOR

- **Taylor Expansion**
  - Wenliang Tseng, ‘Passive Order Reduction for RLC Circuits With Delay Elements’, 2007
  - Taylor expansion of the delay exponential term.
  - Bad approximation.
  - Result in a high-order equivalent LTI system for the Anroldi procedure.

$$x(t-d) \xrightarrow{s-domain} X(s)e^{-sd}$$

$$e^{-sd} = \sum_{i=0}^{\infty} \frac{(-d)^i}{i!} s^i$$

Problem Formulation:
Padé approximation based MOR

- **Padé approximation**
  - Q. Wang, ‘Model Order Reduction For Neutral Systems by Moment Matching’.
  - Padé approx. of the delay exponential term.
  - Bad approximation.
  - Result in a high-order equivalent LTI system for the Anroldi procedure / balanced truncation.

$$x(t-d)\xrightarrow{s-domain} X(s)e^{-sd}$$

$$e^{-sd} \approx \frac{P_d(s)}{Q_d(s)} = \widetilde{C}(sI-\widetilde{A})^{-1}\widetilde{B}+\widetilde{D}$$

Problem Formulation:

MOR via Balanced Truncation

- **Balanced Truncation**
  - Gramians of Time-Delay Systems.
    - *S. Yi, 'Controllability and observability of systems of linear delay differential equations via the matrix Lambert W function', 2008*
  - The state solution involves Lambert W functions.

$$\dot{x}(t) = Ax(t) + A_d x(t-d) + Bu(t), \quad y(t) = Cx(t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{S_k (t-\tau)} C_k^N Bu(\tau) d\tau, \quad where\ S_k = \frac{1}{d} W_k(A_d d Q_k) + A$$

$$P = \int_0^{\infty} \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^N BB^T \left(\sum_{k=-\infty}^{\infty} e^{S_k t} C_k^N\right)^T dt$$

$$Q = \int_0^{\infty} \left(\sum_{k=-\infty}^{\infty} e^{S_k t} C_k^N\right)^T C^T C \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^N dt$$

The Gramians of TDSs

Problem Formulation:

Lyapunov-Type Equations for TDSs

- By defining $$w_{ij} = e^{S_i t} C_i^N BB^T (e^{S_j t} C_j^N)^T$$

$$P_{ij} = \int_0^\infty w_{ij} dt$$

- The controllability Gramian can be rewritten as

$$P = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \int_0^\infty w_{ij} dt = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} P_{ij}$$

- Lyapunov-type equations for the controllability Gramian components

$$S_i P_{ij} + P_{ij} S_j^T + C_i^N BB^T (C_j^N)^T = 0$$

Problem Formulation:
Gramians in the s-domain

- The Lyapunov-type equations are hard to be solved
  - It involves Lambert W functions.
  - Convergence of the branches.
  - High computational cost.

- Gramians in the s-domain *[S. Yi, 2008]*
  - The formulations are much simpler.
  - Use the Poor Man’s TBR to approximate.

$$P = \int_{-\infty}^{\infty} (sI - A - A_d e^{-sd})^{-1} BB^T (sI - A - A_d e^{-sd})^{-H} ds$$

$$Q = \int_{-\infty}^{\infty} (sI - A - A_d e^{-sd})^{-H} C^T C (sI - A - A_d e^{-sd})^{-1} ds$$

Problem Formulation: the Poor Man's TBR

- **The Poor Man's TBR for LTI Systems** *[J. R. Phillips, 2005]*
  - It utilizes Laplace transform of the Gramians.

$$P = \int_{-\infty}^{\infty} (sI - A)^{-1} BB^T (sI - A)^{-H} \, ds$$

$$Q = \int_{-\infty}^{\infty} (sI - A^T)^{-1} C^T C (sI - A^T)^{-H} \, ds$$

  - It uses finite summation to approximate the infinite integration.

$$z_{ck} = (j\omega_k I - A)^{-1} B \qquad z_{ok} = (j\omega_k I - A^T)^{-1} C^T$$

$$\widetilde{P} = \sum_k z_{ck} z_{ck}^H \qquad \widetilde{Q} = \sum_k z_{ok} z_{ok}^H$$

Proposed Algorithm:
TBR using the Approximate Gramians

- **Balanced Truncation for Time-Delay Systems**
  - Calculate the Gramians in the s-domain.

$$P = \int_{-\infty}^{\infty} (sI - A - A_d e^{-sd})^{-1} BB^T (sI - A - A_d e^{-sd})^{-H} ds$$

$$Q = \int_{-\infty}^{\infty} (sI - A - A_d e^{-sd})^{-H} C^T C (sI - A - A_d e^{-sd})^{-1} ds$$

  - Use finite summation to approximate the infinite integration.

$$z_{ck} = (j\omega_k I - A - A_d e^{-j\omega_k d})^{-1} B, \quad z_{ok} = (j\omega_k I - A - A_d e^{-j\omega_k d})^{-H} C^T$$

$$\widetilde{P} = \sum_k z_{ck} z_{ck}^H \qquad \widetilde{Q} = \sum_k z_{ok} z_{ok}^H$$

Proposed Algorithm:
Practical Implementations for TDSs

- **Time-Delay Systems**
  - Multiple delays
  - Descriptor systems

$$E\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_{d_i} x(t - d_i) + Bu(t)$$

$$z_{ck} = (j\omega_k E - A - \sum_{i=1}^{m} A_{d_i} e^{-j\omega_k d_i})^{-1} B$$

$$z_{ok} = (j\omega_k E - A - \sum_{i=1}^{m} A_{d_i} e^{-j\omega_k d_i})^{-H} C^T$$

$$\widetilde{P} = \sum_{k} z_{ck} z_{ck}^H \qquad \widetilde{Q} = \sum_{k} z_{ok} z_{ok}^H$$

Proposed Algorithm:
Balanced Truncation for TDSs

- **The Proposed MOR Procedures for TDSs**
  1. Select the frequency-sampling points uniformly within the frequency band of interest.
  2. Compute the approximate Gramians in the s-domain.
  3. Use canonical TBR to reduce the model.
  4. Compare with moment matching approach and the Padé-based approach.

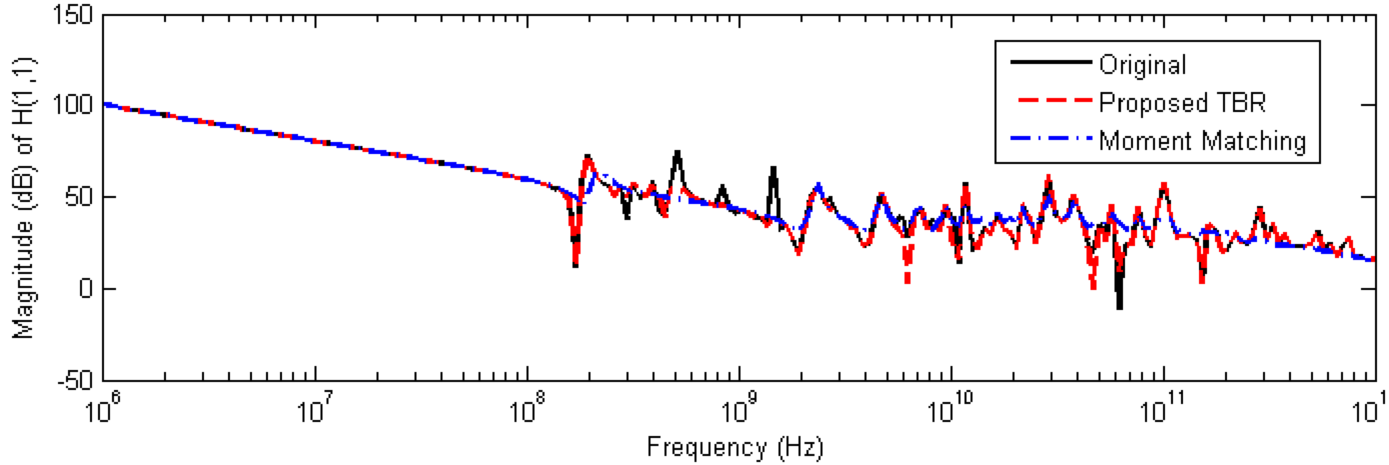
Experimental Results:
A Time-Delay System Example

- **A Time-Delay System Example**
  - A 3-port linear interconnect network.
  - 70 lossless 3-conductor transmission lines.
  - The resulting TDS is of order 1098.
  - The delays are in the magnitude of nanoseconds.
  - The delay effects in the frequency response emerge from around 1GHz.

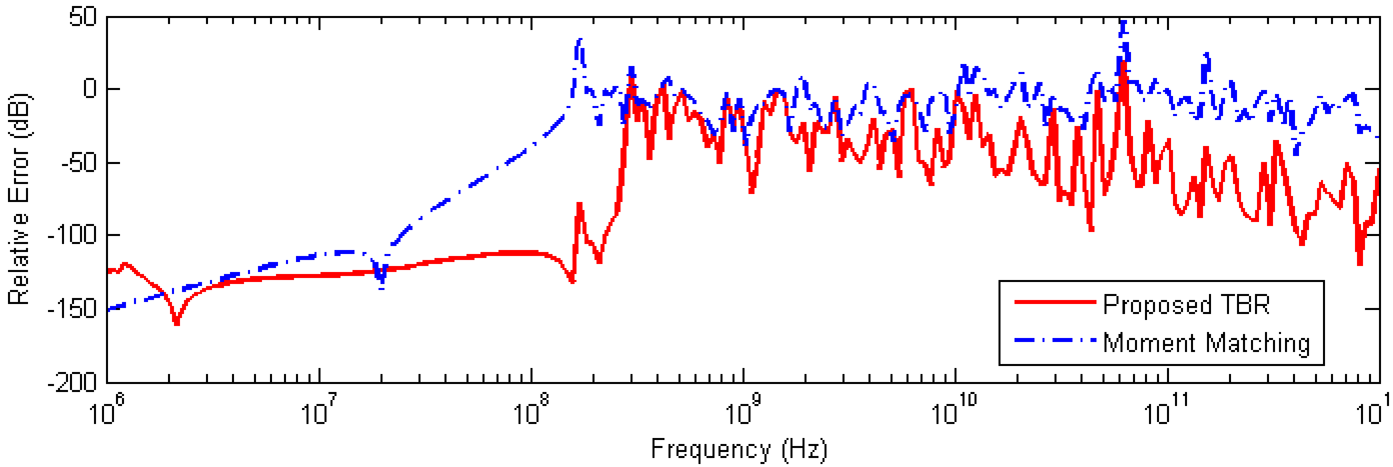
Experimental Results:
A Time-Delay System Example

- **The Reduced-Order Models**
  - By proposed TBR: it’s of order 220.
    - Gramians are approximated by using 50 sampling points distributed uniformly in logarithm scale within the frequency range of interest.
  - By moment matching: it’s of order 231.
    - 2nd-order Taylor expansion.
    - Matching the first 77 moments.
  - By Padé approx. based method: it’s out of memory
    - Only using 2nd-order Padé approximation.

Experimental Results:
Frequency Response

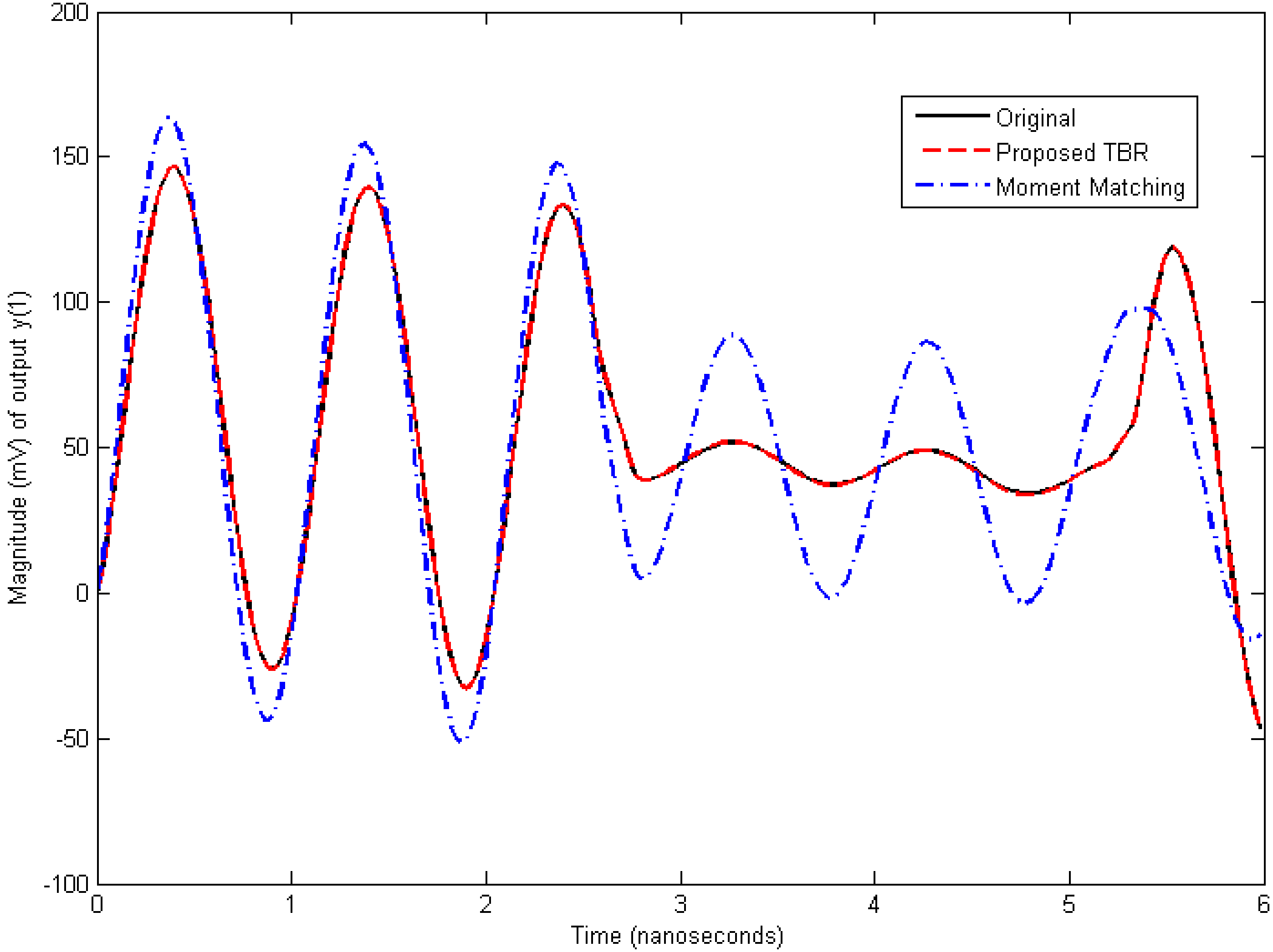


(a)



(b)

Experimental Results:
Time-Domain Response



Experimental Results:
Comparison of Computational Times

- **Speedup of Reduced-Order Models**
  - ROMs by different MORs are of similar order.
  - Around 10X speedup in transient simulation.

- **Building Time of Various MORs**
  - By proposed TBR: 71.07 seconds.
  - By moment matching: 43.51 seconds.
  - By Padé-approx. based method: out of memory.

- **Advantage of the Proposed Algorithm**
  - Higher accuracy with comparable efficiency

Summary and Conclusions

- We proposed and implemented a TBR-type MOR for time-delay systems.
- It's the first time to perform MOR on TDSs via a Gramian-based approach.
- Higher accuracy with comparable efficiency than conventional approaches.

Thanks for your time and attention!