Efficient Matrix Exponential Method Based on Extended Krylov Subspace for Transient Simulation of Large-Scale Linear Circuits

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Outline

- Introduction
  - Circuit Simulation
  - Matrix Exponential Method(MEXP)
- MEXP based on Extended Krylov Subspace
  - Problem of Stiff Circuit
  - Generalized Extended Krylov Subspace
- Numerical Results
- Conclusion

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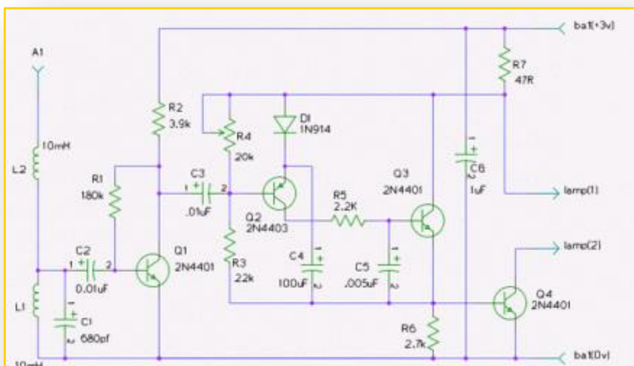
1. Introduction

- 1.1 Circuit Simulation
  - Circuit simulation is to use mathematical models to predict the behavior of an electronic circuit.



Ordinary differential equations (ODEs)

1. Introduction

- 1.2 Matrix Exponential Method (MEXP)
- The numerical system to be solved in transient circuit analysis is a set of differential algebraic equations (DAE)

$$C\dot{x}(t) = Gx(t) + Bu(t)$$

C, G and B: susceptance, conductance and input matrix, respectively

$u(t)$: collects the voltage and current sources

- The essence of MEXP lies in transforming the above equation to an ODE

$$\dot{x}(t) = Ax(t) + b(t)$$

where $A = C^{-1}G$ and $b(t) = C^{-1}Bu(t)$.

$$x(t+h) = e^{Ah}x(t) + \int_0^h e^{A(h-\tau)} b(t+\tau)d\tau$$

piece-wise linear (PWL) input

$$x(t+h) = e^{Ah}x(t) + \left(e^{Ah} - I\right)A^{-1}b(t) + \left(e^{Ah} - (Ah+I)\right)A^{-2}\frac{b(t+h)-b(t)}{h}$$

transform

$$x(t+h) = [I_n \ 0]e^{\tilde{A}h}\begin{bmatrix} x(t) \\ e_2 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} A & W \\ 0 & J \end{bmatrix}, J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, W = \begin{bmatrix} \frac{b(t+h)-b(t)}{h} & b(t) \end{bmatrix}$$

$$\boldsymbol{e^{Ah}}$$

Krylov subspace

For simplicity , we will use A to represent the $\tilde{A}$ in the following part.

1. Introduction

- 1.2 Matrix Exponential Method (MEXP)
  - Main computation is

$$e^{Ah}v \approx \beta V_m e^{\hat{T}_m h} e_1, \qquad \beta = \|v\|_2$$

  - Krylov subspace: $K_m = span\{v, Av, A^2v, \dots A^{m-1}v\}$
  - Arnoldi process: $AV_m = V_{m+1}\hat{T}_m$
    - V_m : orthonormal basis of $K_m(A, v)$
    - $\hat{T}_m$: contains the orthonormalization coefficients
  - Error estimate: $err = \beta t_{m+1,m} \left\| e_m^T e^{T_m h} e_1 \right\|$
    - $t_{m+1,m}$ is the bottom right element of $\hat{T}_m$

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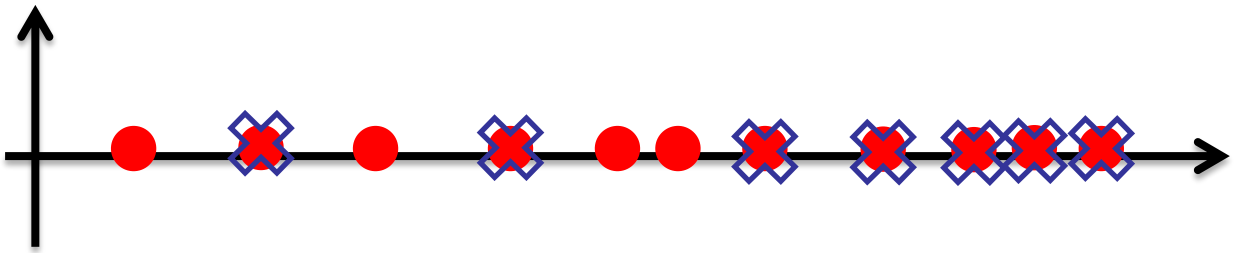
2. MEXP based on Extended Krylov Subspace

- 2.1 Problem for Stiff Circuits
- Stiff circuits:
  - Time constants differ by a large magnitude
  - Real parts of eigenvalues are well-separated
- Shortcomings of Krylov subspace:
  - Tend to capture the dominant eigenvalues first
  - Tend to undersample of small magnitude eigenvalues



2. MEXP based on Extended Krylov Subspace

- 2.1 Problem for Stiff Circuits
- Traditional extended Krylov subspace:
  - Merits: Capture the small magnitude eigenvalues because of the basis vectors from negative power of the matrix
  - Demerits: Computation of negative dimensions are more expensive than the computation of positive dimensions
- Existing extended Krylov subspace:

$$K_{l,m} = span\{A^{-l+1}v, \dots A^{-1}v, v, Av, \dots A^{m-1}v\}$$

$$K_{m,m} = span\{v, A^{-1}v, Av, \dots A^{-m+1}v, A^{m-1}v\}$$

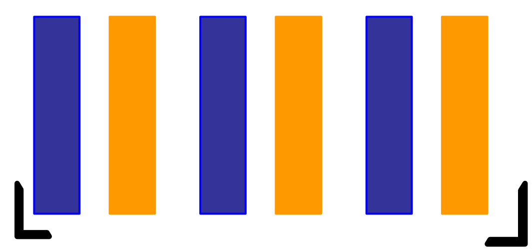
2. MEXP based on Extended Krylov Subspace

- 2.1 Problem of the Stiff Circuit
- Shortcoming of existing extended Krylov subspace:
  - Negative dimension l need to be prespecified, subspace only augments in positive direction

$$K_{l,m} = span\{A^{-l+1}v, \ldots A^{-1}v, v, Av, \ldots A^{m-1}v\}$$

  - Equal number of negative and positive dimension may lead to waste of runtime

$$K_{m,m} = span\{v, A^{-1}v, Av, \ldots A^{-m+1}v, A^{m-1}v\}$$



2. MEXP based on Extended Krylov Subspace

- 2.2 Generalized Extended Krylov Subspace
  - Generalized extended Krylov subspace with ***unequal*** number of positive/negative dimensions:

$$K_{m,km} = span\{v, A^1 v, A^2 v \ldots A^k v, A^{-1} v, A^{k+1} v, \ldots A^{2k} v, A^{-2} v, \ldots, A^{km-1} v, A^{-m+1} v\}$$

  - Arnoldi-type process: $AV_m = V_{m+2}\hat{T}_m$
    - $\hat{T}_m$ is a block Heisenberg matrix
  - Posterior error estimate:

$$err = \beta \tau_{m+1,m} \left\| e_m^T e^{T_m h} e_1 \right\|$$

    - $\tau_{m+1,m}$ is the 2-by-2 bottom right block of $\hat{T}_m$

2. MEXP based on Extended Krylov Subspace

- How to compute $\hat{T}_m$ effectively and economically?
  - From the construction of the generalized extended Krylov subspace, we can get the following recursive relations:

$$
\begin{aligned}
&\text{If } n = 1 \text{ or } \mod (n, k+1) = 2, ..., k \\
&h_{n+1,n} v_{n+1} = A v_n - V_n h_{1:n,n}, \\
&\text{If } \mod (n, k+1) = 0 \\
&h_{n+2,n+1} v_{n+2} = A v_n - V_{n+1} h_{1:n+1,n+1}, \\
&\text{If } n > k \text{ and } \mod (n, k+1) = 1 \\
&h_{n,n-1} v_n = A^{-1} v_{n-k-1} - V_{n-1} h_{1:n-1,n-1}.
\end{aligned}
$$

2. MEXP based on Extended Krylov Subspace

- Can we compute $\hat{T}_m$ without extra matrix-vector products of $V_{m+2}^T A V_m$?

Proposition II.1 *Let* $\hat{T}_m = (t_{i,j})\,, i = 1, ..., 2m+2, j = 1, ..., m$. *Then*

$$
\begin{aligned}
&\text{If } n = 1 \text{ or } \bmod(n, k+1) = 2, ..., k-1 \\
&t_{:,n} = h_{:,n} \\
&\text{If } \bmod(n, k+1) = 0 \\
&t_{:,n} = h_{:,n+1} \\
&\text{If } n > k \text{ and } \bmod(n, k+1) = 1 \\
&t_{:,n} = \frac{1}{h_{n,n-1}} \left(e_{n-k-1} - \begin{bmatrix} \hat{T}_{n-1} h_{1:n-1,n-1} \\ 0 \end{bmatrix} \right)
\end{aligned}
$$

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3. Numerical Results

- 3.1 Improvement led by extended Krylov subspace
- Example: RC ladder
  - Stiff circuit; Matrix order: 1000;
  - Compute $e^{Ah}v$ by four Krylov subspaces
  - Krylov subspace with different negative-positive ratios k=0, 1, 2, 5 (dimension: 24)

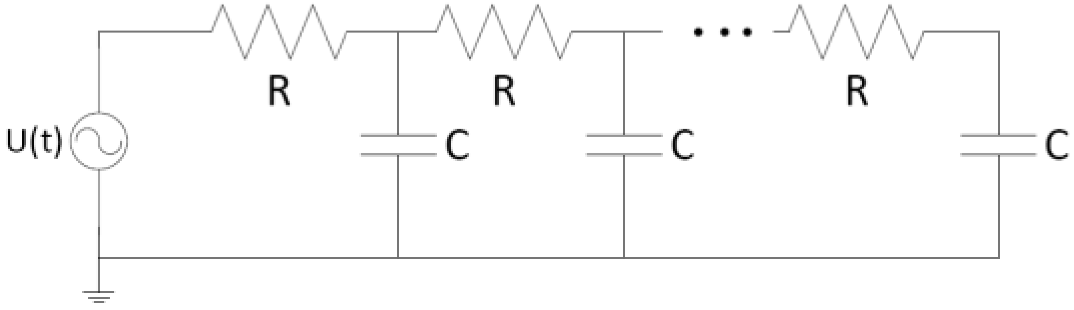

Fig. 1. RC ladder circuit

3. Numerical Results

- 3.1 Improvement led by extended Krylov subspace

TABLE I
ERROR AND RUNTIME OF DIFFERENT KRYLOV SUBSPACES

Subspace	$\mathcal{K}_{24}$	$\mathcal{K}_{12,12}$	$\mathcal{K}_{8,16}$	$\mathcal{K}_{4,20}$
Error	4.2e-1	1.17e-6	3.6e-4	1.4e-2
Time (s)	0.09	0.37	0.17	0.11

- Extended Krylov subspace enjoys higher accuracy but increases runtime as a trade off

3. Numerical Results

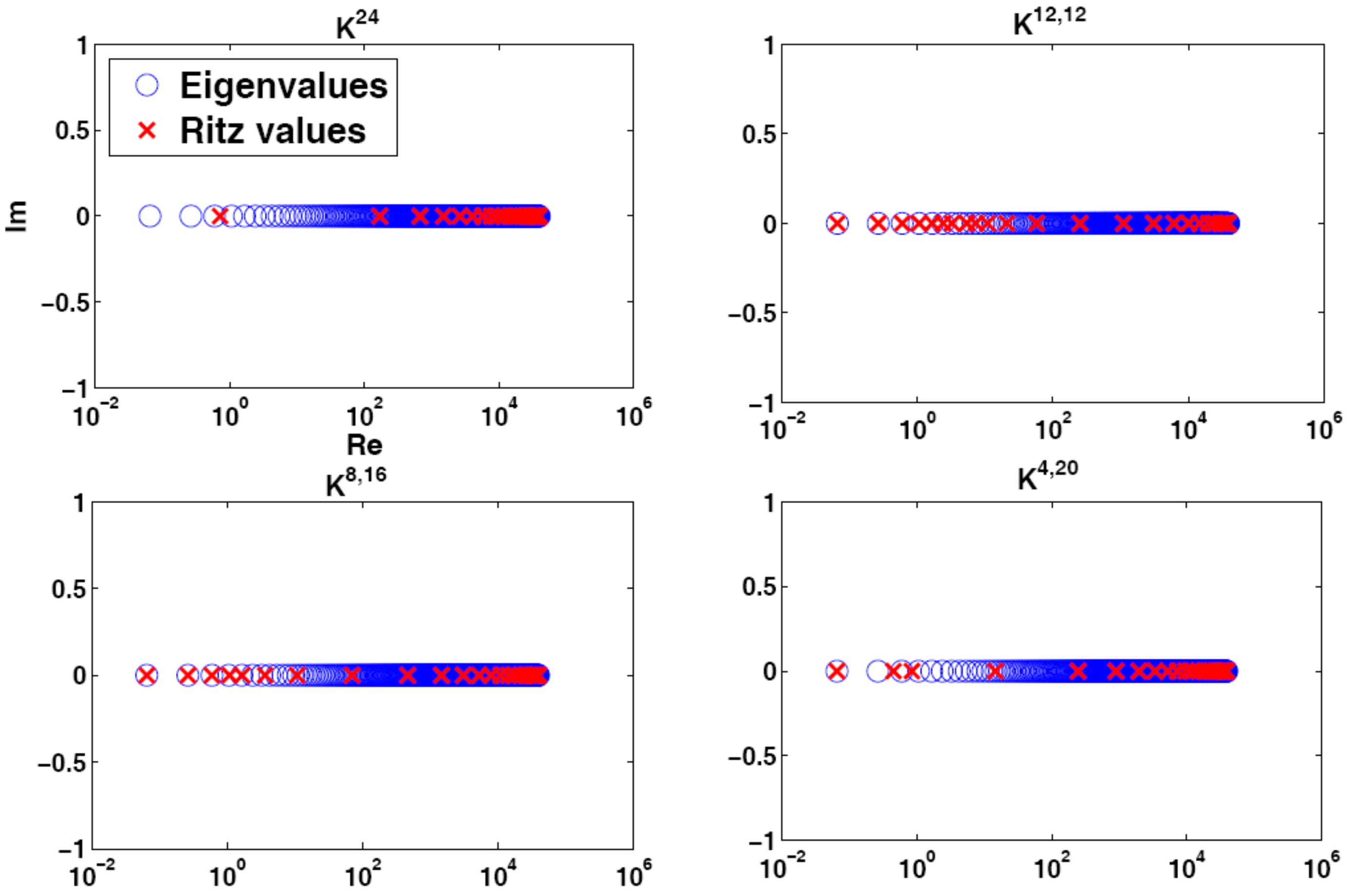

Fig. 2. Approximation of spectrum by standard Krylov subspace and extended Krylov subspace.

3. Numerical Results

- 3.2 Performance of MEXP based on different Krylov subspace with real circuit examples
- Example: three linear circuit examples
  - Run 100 time step with a constant step size
  - Allow the subspace dimension to vary dynamically to satisfy a tolerance of 10^{-6}

TABLE II
SPECIFICATIONS OF TEST CIRCUITS

Circuits	Category	Nodes	Matrix size	Stiffness
C1	Power grid	39K	54K	medium
C2	Power grid	164K	165K	high
C3	Trans. lines	5.6K	8.8K	high

3. Numerical Results

TABLE III
PERFORMANCE OF MEXP BASED ON DIFFERENT KRYLOV SUBSPACE

Circuits	Step size (s)	Subspace dimensions				Total runtime (s)			
		k=0	1	3	4	k=0	1	3	4
C1	1e-12	(0,191)	(35,35)	(29,59)	(16,67)	591.8	641.5	595.1	**390.2**
C2	1e-11	(0,308)	(15,15)	(12,37)	(12,49)	5001.2	1459.1	**1002.7**	1256.3
C3	1e-12	(0,169)	(11,11)	(8,25)	(8,33))	6364.6	**1439.4**	1730.6	2014.7

- Standard Krylov subspace requires a much larger order of the subspace than extended Krylov subspace
- The best breakdown of positive and negative dimensions in extended Krylov subspace is generally problem dependent

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4. Conclusion

- We have investigated the use of extended Krylov subspace to enhance the accuracy of numerical approximation of MEXP-vector product, which in turn benefits the MEXP-based transient circuit simulation.
- We generalize the extended Krylov subspace to allow unequal positive/negative dimensions to maximize the overall performance in circuit simulation.
- Numerical results have confirmed the efficiency of the proposed method.

Q & A

Thank you!