An efficient homomtopy-based Poincaré-Lindstedt method for the periodic steady-state analysis of nonlinear autonomous oscillators

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Motivation

Steady-state analysis of nonlinear autonomous oscillators

- Oscillation frequency unknown: no input
- Shooting Newton and harmonic balancing
  - Dependence initial guess
  - System of nonlinear equations
- Polynomial system solving
  - No dependence on intial guess
  - System of multivariate polynomials
- Poincaré-Lindstedt method
  - Not well-known in EDA
  - No state-space formuation

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Contributions

- state-space formulation Poincaré-Lindstedt method
- homotopy analysis method
- Padé approximation instead of Taylor series

Background

Nonlinear automonous state-space system

$$\dot{x} = G\,x + \mu\,f(x),$$

with $\mu \in \mathbb{R}, x \in \mathbb{R}^n, G, A_k \in \mathbb{R}^{n \times n^k}$ $(k = 1, \ldots, d)$ and

$$f(x) = A_1\,x + A_2\,x \otimes x + A_3\,x \otimes x \otimes x + \cdots\,.$$

$\otimes$ denotes the Kronecker product, e.g. $n = 2$

$$x \otimes x = \begin{pmatrix} x_1^2 & x_1 x_2 & x_2 x_1 & x_2^2 \end{pmatrix}^T.$$

Duffing oscillator

$$\ddot{y} + y + \mu y^3 = 0$$

$$\Rightarrow x = \begin{pmatrix} y & \dot{y} \end{pmatrix}^T, A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, A_3(2,1) = -\mu.$$

Taylor series fail

Suppose $x(t, \mu)$ is periodic in t, then

$$x(t, \mu) = x_0(t) + x_1(t)\,\mu + x_2(t)\,\mu^2 + \cdots \tag{1}$$

Fix degree k of Taylor series, plug (1) into state-space system and solve for $x_0(t), x_1(t), \ldots, x_k(t)$.

Taylor series fail

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Fix degree k of Taylor series, plug (1) into state-space system and solve for $x_0(t), x_1(t), \ldots, x_k(t)$. This fails in practice, e.g. the Duffing oscillator

$$x_1(t) = \begin{pmatrix} -\frac{3}{8}t\sin(t) - \frac{1}{32}(\cos(t) - \cos(3t)) \\ -\frac{3}{8}t\cos(t) - \frac{1}{32}(11\sin(t) + 3\sin(3t)) \end{pmatrix},$$

which contains secular terms $t\sin(t), t\cos(t)$ that grow unbounded with t.

Poincaré-Lindstedt method

Introduce new μ-dependent time parameter τ

$$\tau = \omega(\mu)\, t.$$

The $\omega(\mu)$ parameter stretches the time-scale according to the fixed value of μ. We can model $\omega(\mu)$ as

$$\omega(\mu) = \omega_0 + \omega_1 \mu + \omega_2 \mu^2 + \cdots$$

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With a new time parameter comes a new state vector $z(\tau, \mu) := x(t, \mu)$

$$\frac{dz}{dt} = \frac{dz}{d\tau}\,\frac{d\tau}{dt} = \omega\,\frac{dz}{d\tau} \Rightarrow \omega\,\frac{dz}{d\tau} = G\,z + \mu\,f(z).$$

Additional assumptions

$$z(\tau,\mu) = z_0(\tau) + z_1(\tau)\,\mu + z_2(\tau)\,\mu^2 + \cdots .$$

Impose additional periodicity on $z(\tau,\mu)$, e.g.
$z(\tau,\mu) = z(\tau + 2\pi,\mu)$ for all $\mu > O$, which implies that

$$z_k(\tau,\mu) = z_k(\tau + 2\pi,\mu), \quad k = 0, 1, 2, \ldots ,$$

which removes the secular terms.

Solution strategy

1. Substitute Taylor series for $\omega(\mu)$ and $z(\tau,\mu)$ into $\omega\,\frac{dz}{d\tau} = G\,z + \mu\,f(z)$,

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2. Equating coefficients of equal degrees of μ leads to the subsystems,

$$\begin{aligned}
\omega_0\,\frac{dz_0}{d\tau} &= G\,z_0,\\
\omega_0\,\frac{dz_1}{d\tau} + \omega_1\,\frac{dz_0}{d\tau} &= G\,z_1 + f_1(z_0),\\
\omega_0\,\frac{dz_2}{d\tau} + \omega_1\,\frac{dz_1}{d\tau} + \omega_2\,\frac{dz_0}{d\tau} &= G\,z_2 + f_2(z_0,z_1),\\
&\vdots
\end{aligned}$$

with $f_1(z_0), f_2(z_0,z_1),\ldots$ polynomial functions in $z_0, z_1, \ldots$.

Solution strategy continued

3. Solve LTI system $\omega_0 \frac{dz_0}{d\tau} = G\, z_0$ for $z_0(\tau)$,
4. Determine ω_0 from $z_0(\tau, \mu) = z_0(\tau + 2\pi, \mu)$,
5. Solve LTI system $\omega_0 \frac{dz_1}{d\tau} + \omega_1 \frac{dz_0}{d\tau} = G\, z_1 + f_1(z_0)$ for $z_1(\tau)$,
6. Determine ω_1 from $z_1(\tau, \mu) = z_1(\tau + 2\pi, \mu)$,
7. etc....

Solution strategy continued

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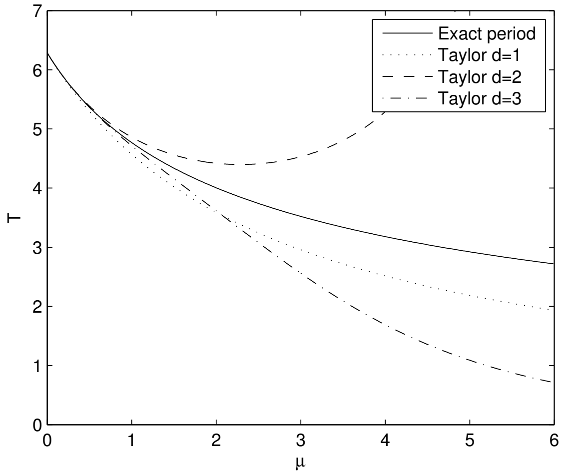
6. Determine ω_1 from $z_1(\tau, \mu) = z_1(\tau + 2\pi, \mu)$,
7. etc....

New problem

Only works for values of μ close to 0.

Taylor series approximation for period of Duffing oscillator



State-space Homotopy Method

Establish the following homotopy mapping $\Psi(\tau, p) \in \mathbb{R}^n$ from the initial solution $\Psi(\tau, 0) := z(\tau, 0)$ to $\Psi(\tau, 1) := z(\tau, \mu)$:

$$(1-p)\left[\frac{d}{d\tau}\Psi - G\Psi\right] = hp\left[u(p)\frac{d}{d\tau}\Psi - G\Psi - \mu f(\Psi)\right],$$

- $\Psi(\tau, p) = \Psi(\tau + 2\pi, p)$ for any $p \in [0, 1]$,
- $\Psi(0, p) = a(p)$,
- h is a nonzero auxiliary parameter,
- $u(1) = \omega(\mu)$ and $a(1) = z(0, \mu)$.

Additional assumptions

$$\Psi(\tau, p) = \sum_{k=0}^{\infty} \Psi_k(\tau)\, p^k,$$

$$u(p) = \sum_{k=0}^{\infty} u_k(\tau)\, p^k,$$

$$a(p) = \sum_{k=0}^{\infty} a_k(\tau)\, p^k,$$

with $\Psi_k(\tau) = \Psi_k(\tau + 2\pi)$ and $\Psi_k(0) = a_k$.

New solution strategy

1. Substitute Taylor series for $\Psi(\tau, p), u(p), a(p)$ into
$(1-p)\ \left[\frac{d}{d\tau}\Psi - G\Psi\right] = hp\left[u(p)\frac{d}{d\tau}\Psi - G\Psi - \mu f(\Psi)\right]$,

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2. Equating coefficients of equal degrees of p leads to the subsystems,

$$\frac{d\Psi_0}{d\tau} = G\,\Psi_0,$$

$$\left(\frac{d\Psi_1}{d\tau} - G\,\Psi_1\right) - \left(\frac{d\Psi_0}{d\tau} - G\,\Psi_0\right) = h\left[u_0\frac{d\Psi_0}{d\tau} - G\,\Psi_0 - \mu f_1(\Psi_0)\right],$$

$$\left(\frac{d\Psi_2}{d\tau} - G\,\Psi_2\right) - \left(\frac{d\Psi_1}{d\tau} - G\,\Psi_1\right) =$$

$$h\left(u_0\frac{d\Psi_1}{d\tau} + u_1\frac{d\Psi_0}{d\tau} - G\,\Psi_1 - \mu f_2(\Psi_0, \Psi_1)\right),$$

$$\vdots$$

Solution strategy continued

3. Solve LTI system $\frac{d\Psi_0}{d\tau} = G\,\Psi_0$,
4. Determine u_0, a_0 from $\Psi_0(\tau,\mu) = \Psi_0(\tau + 2\pi, \mu)$,
5. Solve system
   $(\frac{d\Psi_1}{d\tau} - G\,\Psi_1) - (\frac{d\Psi_0}{d\tau} - G\,\Psi_0) = h\left[u_0\frac{d\Psi_0}{d\tau} - G\,\Psi_0 - \mu f_1(\Psi_0)\right]$
   for $\Psi_1(\tau)$,
6. Determine u_1, a_1 from $\Psi_1(\tau,\mu) = \Psi_1(\tau + 2\pi, \mu)$,
7. etc....

Solution strategy continued

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   for $\Psi_1(\tau)$,
6. Determine u_1, a_1 from $\Psi_1(\tau,\mu) = \Psi_1(\tau + 2\pi, \mu)$,
7. etc....

Order of computation

$$\Psi_0 \rightarrow u_0, a_0 \rightarrow \Psi_1 \rightarrow u_1, a_1 \rightarrow \cdots$$

Computed results

$$z(\tau, \mu) = \sum_{k=0}^{d} \Psi_k,$$

$$\omega(\mu) = \sum_{k=0}^{d} u_k,$$

$$z(0, \mu) = \sum_{k=0}^{d} a_k.$$

Computed results

$$z(\tau, \mu) = \sum_{k=0}^{d} \Psi_k,$$

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Better period estimation

Use Padé approximant for $u(p)$, can be computed directly from the Taylor series.

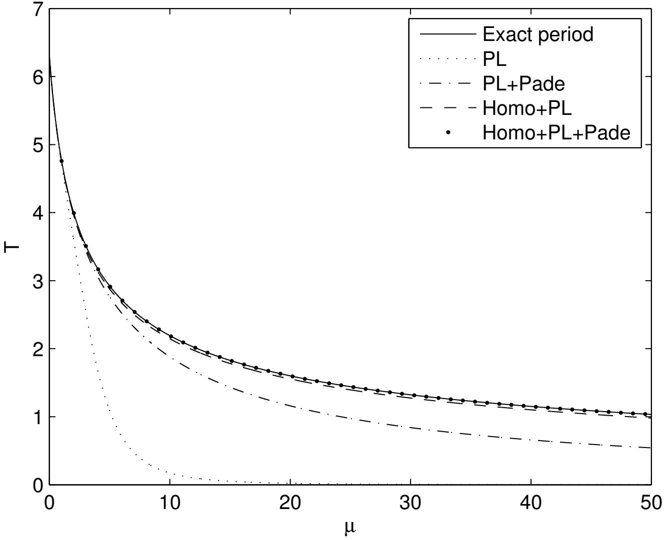
Duffing oscillator

Maximal μ such that $\frac{|T_{\text{exact}}(\mu)-T_{\text{approx}}(\mu)|}{|T_{\text{exact}}(\mu)|} \leq 5\%$,

Table : Trust region of μ with a relative period error of 5%

	PL	PL+Pade	Homo+PL	Homo+PL+Pade
μ	(0, 1.60]	(0, 4.77]	(0, 48.14]	(0, >500]

Duffing oscillator



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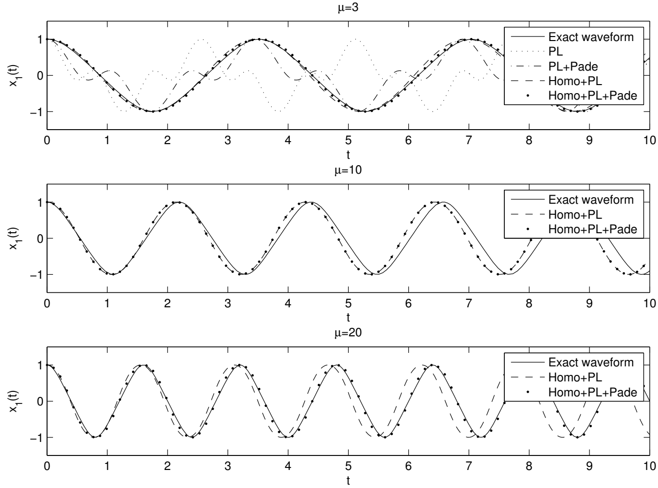

Conclusions

- Original nonlinear differential equation is divided into subproblems.
- Each subproblem does not increase the scale of the original problem.
- Frequency is not required to be known a priori.
- Padé approximation enhances accuracy of the period estimation.

Thank you!